Quantum State Discrimination task- Some recent results

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- Mainly Single copy case and with certainty.
- The class of Bell states and generalized Bell states.
- Next, we will discuss about a class of activable bound entangled states.
- Lastly, we will provide some recent results we got regarding indistinguishability and distinguishability of certain classes of states and related topics.

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- A particular type of physical operation is very much useful in quantum information theory, viz., the separable superoperator.

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- Suppose a quantum state is shared between a number of parties, say, A, B, C, D, etc., and each A_k in the Krause representation has the form $A_k = L_k^A \otimes L_k^B \otimes L_k^C \otimes L_k^D \otimes \cdots$ where all $L_k^A, L_k^B, L_k^C, L_k^D, \cdots$ are linear operators,

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- then the map $\mathcal{E}$ is usually called a separable superoperator.
- By LOCC, we mean a physical operation where each party can perform quantum operations in their subsystems and communicate their results to the others and this process could be performed in infinitely many rounds.
- It is interesting to note that every LOCC is a separable superoperator but the converse is not always true.

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- A complete orthogonal basis of 3x3 system, $|0\rangle|0\pm1\rangle, |2\rangle|1\pm2\rangle, |1\pm2\rangle|0\rangle, |0\pm1\rangle|2\rangle, |1\rangle|1\rangle$, are not locally distinguishable with certainty in the single copy case.

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- Another most striking example of an orthogonal product basis is the discovery of Unextendible product basis by Bennett et. al.[PRL, 82(1999),5385], $|0\rangle|0-1\rangle, |2\rangle|1-2\rangle, |1-2\rangle|0\rangle, |0-1\rangle|2\rangle, |0+1+2\rangle|0+1+2\rangle$ in 3x3.This set is not exactly locally distinguishable. Later many classes of UPB have been formed, and all have the same property regarding distinguishability.

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- Quite contrary to the above cases, it is found by Hardy, Walgate, [PRL, 85 (2000), 4972], that any two orthogonal pure states of bipartite or multipartite systems(whether entangled or not) are locally distinguishable with certainty. They have shown that any two orthogonal pure bipartite states can be expressed as;

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- $|\psi\rangle = |0\rangle|\chi\rangle + |1\rangle|\xi\rangle + |2\rangle|\eta\rangle + \ldots.$
- $|\phi\rangle = |0\rangle|\overline{\chi}\rangle + |1\rangle|\overline{\xi}\rangle + |2\rangle|\overline{\eta}\rangle + \ldots.$
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  are orthogonal to each other for the first system and $|\chi\rangle, |\overline{\chi}\rangle$, $|\xi\rangle, |\overline{\xi}\rangle$, $|\eta\rangle, |\overline{\eta}\rangle \cdots$ are pairwise orthogonal.
- Consider now four Bell states, $|\Phi^{\pm}\rangle \equiv \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $|\Psi^{\pm}\rangle \equiv \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$. They are not locally distinguishable with certainty in the single copy case [PRL, 87 (2001),277902]. The proof is quite interesting.

Smolin state and local indistinguishability of Bell states

- Consider the four qubit states shared among four distant parties: $\rho_4^+ = \frac{1}{4}\{P[\Phi^+]\otimes P[\Phi^+] + P[\Phi^-]\otimes P[\Phi^-] + P[\Psi^+]\otimes P[\Psi^+] + P[\Psi^-]\otimes P[\Psi^-]\}$

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- where $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are the Bell states, written in their usual basis and $P[\cdot]$ represents projectors on those states. The state ρ_4^+, known as Smolin state or unlockable/activable bound entangled state [Phys. Rev. A **63**, 032306 (2001)].

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- The state is symmetric w.r.t. all the four parties, separable in 2:2 cut and if any two parties came together in a lab, then it is possible to share a Bell state among other two parties.
- If Bell states are locally distinguishable, then by LOCC only (not coming together in a lab) any two parties can distinguish which Bell state they are sharing and correspondingly other two parties will share a definite Bell state. As the state is separable in any 2:2 cut, it is not possible to share Bell states to other two parties situating all the four in distant labs.

Further results

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- If at least one of the states is entangled then the basis is not locally distinguishable and the basis is probabilistically distinguishable iff all are product.

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- $|\psi_{mn}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left(\frac{2\pi ijn}{d}\right) |j\rangle\, |j + m \mod d\rangle$, where $n, m = 0, 1, ..., d - 1$.

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- All are connected with $|\psi_{00}\rangle$ locally unitarily by $U_{nm} \otimes I$, where, $U_{nm} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left(\frac{2\pi ijn}{d}\right) |j\rangle \langle j + m \bmod d|$, are trace orthogonal.

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- The class of maximally entangled states are also known as generalized Bell states.

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- Later, Nathanson [JMP, 46(2005), 062103] generalized this result to any set of three maximally orthogonal states $C^3 \otimes C^3$. Using the existence of mutually unbiased basis he also found the condition for a set of k maximally entangled states. Result of Fan is reproduced and also established the result for $d+1$ states.

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- As the protocol is fixed, therefore the qudits obtained after teleportation are different.

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- Two observations: if two copies of each generalized Bell states are supplied then by teleportation protocol it is shown that the full set of d^2 states are locally distinguishable.

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- Two observations: if two copies of each generalized Bell states are supplied then by teleportation protocol it is shown that the full set of d^2 states are locally distinguishable.
- If a set of d or less than d states are locally distinguishable, then considering the teleported states as basis, we can rearrange the states in such a manner so that they could be distinguishable by 1-way LOCC.

Contd..

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- $|\psi_{10}\rangle = \frac{1}{\sqrt{6}}[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \cdots \omega^5|5\rangle|5\rangle]$
- $|\psi_{30}\rangle = \frac{1}{\sqrt{6}}[|0\rangle|0\rangle + \omega^3|1\rangle|1\rangle + \cdots \omega^3|5\rangle|5\rangle]$
- $|\psi_{03}\rangle = \frac{1}{\sqrt{6}}[|0\rangle|3\rangle + |1\rangle|4\rangle + \cdots |5\rangle|2\rangle]$
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- Teleportation protocol fails for this set of states. Also, in 4x4, 5x5, there are examples of four states where the protocol fails.
- Later, Ghosh et al and Bandyopadhyay et al provide other conditions by which the problem of local discrimination of the class of *d* maximally entangled states by 1-way LOCC resolved.

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- where $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$ are the Bell states, written in their usual basis and $P[\cdot]$ represents projectors on those states. The state ρ_4^{+}, is the Smolin state described earlier.
- The class of activable bound entangled states in multiqubit systems [Phys. Rev. A **71**, 062317 (2005)] constructed as follows:

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- If we denote the $2N$ qubit states as $\rho^{\pm}_{2N}$, $\sigma^{\pm}_{2N}$ then the next four states of $2N+2$ qubit system are given by,

$$\begin{aligned} \rho^{\pm}_{2N+2} &= \tfrac{1}{4}\,\{\rho^{+}_{2N} \otimes P[\Phi^{\pm}] + \rho^{-}_{2N} \otimes P[\Phi^{\mp}] + \sigma^{+}_{2N} \otimes P[\Psi^{\pm}] \\ &\quad + \sigma^{-}_{2N} \otimes P[\Psi^{\mp}]\} \\ \sigma^{\pm}_{2N+2} &= \tfrac{1}{4}\,\{\rho^{+}_{2N} \otimes P[\Psi^{\pm}] + \rho^{-}_{2N} \otimes P[\Psi^{\mp}] + \sigma^{+}_{2N} \otimes P[\Phi^{\pm}] \\ &\quad + \sigma^{-}_{2N} \otimes P[\Phi^{\mp}]\} \end{aligned} \tag{2}$$

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- If we denote the $2N$ qubit states as $\rho^{\pm}_{2N}$, $\sigma^{\pm}_{2N}$ then the next four states of $2N+2$ qubit system are given by,

$$\begin{aligned}
\rho^{\pm}_{2N+2} &= \frac{1}{4}\{\rho^{+}_{2N} \otimes P[\Phi^{\pm}] + \rho^{-}_{2N} \otimes P[\Phi^{\mp}] + \sigma^{+}_{2N} \otimes P[\Psi^{\pm}] \\
&\quad +\sigma^{-}_{2N} \otimes P[\Psi^{\mp}]\} \\
\sigma^{\pm}_{2N+2} &= \frac{1}{4}\{\rho^{+}_{2N} \otimes P[\Psi^{\pm}] + \rho^{-}_{2N} \otimes P[\Psi^{\mp}] + \sigma^{+}_{2N} \otimes P[\Phi^{\pm}] \\
&\quad +\sigma^{-}_{2N} \otimes P[\Phi^{\mp}]\}
\end{aligned} \tag{2}$$

- The above formula enables one to generate the whole class of states from the four qubit states by a recursive process. We now show some special features of this class of states.

Contd..

- **Permutation Symmetry:** The whole class of states are symmetric over all the parties concerned, i.e., the states remain invariant under the interchange of any two parties.

Contd..

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- **Orthogonality:** From Eq.(2) the four states of $2N+2$ qubit system are orthogonal to each other if the $2N$ qubit states are so. Also from Eq.(1) we observe the four states $\rho_4^{\pm}$, $\sigma_4^{\pm}$ are mutually orthogonal. Thus recursively it provides orthogonality of the four activable bound entangled states of any even qubit systems.

Contd..

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- **Orthogonality:** From Eq.(2) the four states of $2N+2$ qubit system are orthogonal to each other if the $2N$ qubit states are so. Also from Eq.(1) we observe the four states $\rho_4^{\pm}$, $\sigma_4^{\pm}$ are mutually orthogonal. Thus recursively it provides orthogonality of the four activable bound entangled states of any even qubit systems.
- **Local Indistinguishability:** The four states of $2N$ qubit system, for $N \geq 2$ are locally indistinguishable. The proof is also done recursively. The argument is mainly based on generation of entanglement between any two parties if other two parties are able to distinguish the class of four states locally, which is impossible as parties are in distant labs and states are separable in any 2:2 cut.

Nonlocality without entanglement

- We have discussed mainly several results on distinguishability or indistinguishability of class of orthogonal bipartite pure states and especially the problem of local discrimination of maximally entangled states in $d \times d$. Several other results also provided by many groups. Recently, entanglement assisted discrimination protocols generates further interests on many class of states which initially are locally indistinguishable.

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- Looking back to the result of nonlocality without entanglement where Bennett et al provided full class of product states and a set of UPB in 3x3. In multipartite systems, this concept is still studied incompletely except for some special completely orthogonal product bases(COPB) and some unextendible product bases(UPBs).

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- Looking back to the result of nonlocality without entanglement where Bennett et al provided full class of product states and a set of UPB in 3x3. In multipartite systems, this concept is still studied incompletely except for some special completely orthogonal product bases(COPB) and some unextendible product bases(UPBs).
- We now provide a result in tripartite system which exhibits nonlocality without entanglement but are distinguishable if we provide some assistance by entanglement.

Some LOCC indistinguishable classes of product states in tripartite system

Example 1: In $\mathbb{C}^6 \bigotimes \mathbb{C}^6 \bigotimes \mathbb{C}^6$ the set of 36 orthogonal product states

$$\left\{\begin{array}{lll}
|2+3\rangle\,|0\rangle\,|3\rangle & |2+3\rangle\,|2\rangle\,|5\rangle & |4\pm5\rangle\,|2\rangle\,|5\rangle \\
|3\rangle\,|2+3\rangle\,|0\rangle & |5\rangle\,|2+3\rangle\,|2\rangle & |5\rangle\,|4\pm5\rangle\,|2\rangle \\
|0\rangle\,|3\rangle\,|2+3\rangle & |2\rangle\,|5\rangle\,|2+3\rangle & |2\rangle\,|5\rangle\,|4\pm5\rangle \\
 & & \\
|2\pm3\rangle\,|2\rangle\,|3\rangle & |3\pm4\rangle\,|2\rangle\,|4\rangle & \\
|3\rangle\,|2\pm3\rangle\,|2\rangle & |4\rangle\,|3\pm4\rangle\,|2\rangle & \\
|2\rangle\,|3\rangle\,|2\pm3\rangle & |2\rangle\,|4\rangle\,|3\pm4\rangle & \\
 & & \\
|1\pm2\rangle\,|1\rangle\,|3\rangle & |0\pm1\rangle\,|0\rangle\,|3\rangle & \\
|3\rangle\,|1\pm2\rangle\,|1\rangle & |3\rangle\,|0\pm1\rangle\,|0\rangle & \\
|1\rangle\,|3\rangle\,|1\pm2\rangle & |0\rangle\,|3\rangle\,|0\pm1\rangle &
\end{array}\right\}$$

is LOCC indistinguishable.

Contd..

Theorem 1: In $\mathbb{C}^{2d}\bigotimes\mathbb{C}^{2d}\bigotimes\mathbb{C}^{2d}$, where d is odd, the set of $18(d-1)$ orthogonal product states

$\left|\phi^{\pm}_{i+1}\right\rangle=\left|i\pm\overline{i+1}\right\rangle|i\rangle|d\rangle,\ i=0,1,2,...,d-1.$

$\left|\phi^{\pm}_{d+1+i}\right\rangle=|d\rangle\left|i\pm\overline{i+1}\right\rangle|i\rangle,\ i=0,1,2,...,d-1.$

$\left|\phi^{\pm}_{2d+1+i}\right\rangle=|i\rangle|d\rangle\left|i\pm\overline{i+1}\right\rangle,\ i=0,1,2,...,d-1.$

$\left|\phi^{\pm}_{2d+1+i}\right\rangle=\left|i\pm\overline{i+1}\right\rangle|d-1\rangle|i+1\rangle,$
$i=d,d+1,d+2,...,2d-2.$

$\left|\phi^{\pm}_{3d+i}\right\rangle=|i+1\rangle\left|i\pm\overline{i+1}\right\rangle|d-1\rangle,\ i=d,d+1,d+2,...,2d-2.$

$\left|\phi^{\pm}_{4d-1+i}\right\rangle=|d-1\rangle|i+1\rangle\left|i\pm\overline{i+1}\right\rangle,$
$i=d,d+1,d+2,...,2d-2.$

$\left|\phi_{6d-2+\frac{i}{2}}\right\rangle=\left|\overline{d-1}+d\right\rangle|i\rangle|d\rangle,\ i=0,2,4,...,d-3.$

$\left|\phi_{6d-1+\frac{d-3}{2}+\frac{i}{2}}\right\rangle=|d\rangle\left|\overline{d-1}+d\right\rangle|i\rangle,\ i=0,2,4,...,d-3.$

$\left|\phi_{7d-3+\frac{i}{2}}\right\rangle=|i\rangle|d\rangle\left|\overline{d-1}+d\right\rangle,\ i=0,2,4,...,d-3.$

Contd..

$$\left|\psi_{7d-2+\frac{d-3}{2}+\frac{i-1}{2}}\right\rangle = \left|\overline{d-2}+\overline{d-1}\right\rangle |i\rangle |d\rangle,\ i = 1,3,5,...,d-4.$$
$$\left|\psi_{8d-5+\frac{i-1}{2}}\right\rangle = |d\rangle \left|\overline{d-2}+\overline{d-1}\right\rangle |i\rangle,\ i = 1,3,5,...,d-4.$$
$$\left|\psi_{8d-5+\frac{d-3}{2}+\frac{i-1}{2}}\right\rangle = |i\rangle |d\rangle \left|\overline{d-2}+\overline{d-1}\right\rangle,\ i = 1,3,5,...,d-4.$$
$$\left|\psi_{9d-9+\frac{i-d}{2}}\right\rangle = \left|\overline{d-1}+d\right\rangle |d-1\rangle |i\rangle,\ i = d+2,d+4,...,2d-1.$$
$$\left|\psi_{9d-9+\frac{i-1}{2}}\right\rangle = |i\rangle \left|\overline{d-1}+d\right\rangle |d-1\rangle,\ i = d+2,d+4,...,2d-1.$$
$$\left|\psi_{10d-10+\frac{i-d}{2}}\right\rangle = |d-1\rangle |i\rangle \left|\overline{d-1}+d\right\rangle,$$
$i = d+2, d+4, ..., 2d-1.$
$$\left|\psi_{10d-11+\frac{i}{2}}\right\rangle = \left|d+\overline{d+1}\right\rangle |d-1\rangle |i\rangle,\ i = d+3,d+5,...,2d-2.$$
$$\left|\psi_{11d-12+\frac{i-d-1}{2}}\right\rangle = |i\rangle \left|d+\overline{d+1}\right\rangle |d-1\rangle,$$
$i = d+3, d+5, ..., 2d-2.$

cannot be perfectly distinguished by LOCC.

Contd..

Theorem 2: In $\mathbb{C}^{2d} \bigotimes \mathbb{C}^{2d} \bigotimes \mathbb{C}^{2d}$, where d is even, the set of $18(d-1)$ orthogonal product states

$\left|\phi^{\pm}_{i+1}\right\rangle = \left|i \pm \overline{i+1}\right\rangle |i\rangle |d\rangle,\ i = 0, 1, 2, ..., d-1.$

$\left|\phi^{\pm}_{d+1+i}\right\rangle = |d\rangle \left|i \pm \overline{i+1}\right\rangle |i\rangle,\ i = 0, 1, 2, ..., d-1.$

$\left|\phi^{\pm}_{2d+1+i}\right\rangle = |i\rangle |d\rangle \left|i \pm \overline{i+1}\right\rangle,\ i = 0, 1, 2, ..., d-1.$

$\left|\phi^{\pm}_{2d+1+i}\right\rangle = \left|i \pm \overline{i+1}\right\rangle |d-1\rangle |i+1\rangle,$
$i = d, d+1, d+2, ..., 2d-2.$

$\left|\phi^{\pm}_{3d+i}\right\rangle = |i+1\rangle \left|i \pm \overline{i+1}\right\rangle |d-1\rangle,\ i = d, d+1, d+2, ..., 2d-2.$

$\left|\phi^{\pm}_{4d-1+i}\right\rangle = |d-1\rangle |i+1\rangle \left|i \pm \overline{i+1}\right\rangle,$
$i = d, d+1, d+2, ..., 2d-2.$

$\left|\phi_{6d-2+\frac{i-1}{2}}\right\rangle = \left|\overline{d-1} + d\right\rangle |i\rangle |d\rangle,\ i = 1, 3, 5, ..., d-3.$

$\left|\phi_{6d-1+\frac{d-4}{2}+\frac{i-1}{2}}\right\rangle = |d\rangle \left|\overline{d-1} + d\right\rangle |i\rangle,\ i = 1, 3, 5, ..., d-3.$

Contd..

$$\left|\phi_{7d-4+\frac{i-1}{2}}\right\rangle = |i\rangle\,|d\rangle\,\left|\overline{d-1}+d\right\rangle,\ i=1,3,5,...,d-3.$$

$$\left|\psi_{7d-3+\frac{d-4}{2}+\frac{i}{2}}\right\rangle = \left|\overline{d-2}+\overline{d-1}\right\rangle|i\rangle\,|d\rangle,\ i=0,2,4,...,d-4.$$

$$\left|\psi_{8d-6+\frac{i}{2}}\right\rangle = |d\rangle\left|\overline{d-2}+\overline{d-1}\right\rangle|i\rangle,\ i=0,2,4,...,d-4.$$

$$\left|\psi_{8d-5+\frac{d-4}{2}+\frac{i}{2}}\right\rangle = |i\rangle\,|d\rangle\left|\overline{d-2}+\overline{d-1}\right\rangle,\ i=0,2,4,...,d-4.$$

$$\left|\psi_{9d-9+\frac{i-d}{2}}\right\rangle = \left|\overline{d-1}+d\right\rangle|d-1\rangle\,|i\rangle,\ i=d+2,d+4,...,2d-2.$$

$$\left|\psi_{9d-10+\frac{i}{2}}\right\rangle = |i\rangle\left|\overline{d-1}+d\right\rangle|d-1\rangle,\ i=d+2,d+4,...,2d-2.$$

$$\left|\psi_{10d-11+\frac{i-d}{2}}\right\rangle = |d-1\rangle\,|i\rangle\left|\overline{d-1}+d\right\rangle,$$
$$i=d+2,d+4,...,2d-2.$$

$$\left|\psi_{10d-11+\frac{i-3}{2}}\right\rangle = \left|d+\overline{d+1}\right\rangle|d-1\rangle\,|i\rangle,$$
$$i=d+3,d+5,...,2d-1.$$

$$\left|\psi_{11d-13+\frac{i-d-1}{2}}\right\rangle = |i\rangle\left|d+\overline{d+1}\right\rangle|d-1\rangle,$$
$$i=d+3,d+5,...,2d-1.$$

Theorem 3: In $\mathbb{C}^{2k+1} \bigotimes \mathbb{C}^{2l+1} \bigotimes \mathbb{C}^{2m+1}$, the set of $6(k+l+m)-5$ orthogonal product states

$|\phi_{i,i+1}\rangle = \left|i \pm \overline{i+1}\right\rangle |2l\rangle |m\rangle$, $i = 1, 3, 5, ..., (2k-1)$.

$|\phi_{2k+i,2k+i+1}\rangle = |2k\rangle |l\rangle \left|i \pm \overline{i+1}\right\rangle$, $i = 1, 3, 5, ..., (2m-1)$.

$|\phi_{2k+2m+i,2k+2m+i+1}\rangle = |k\rangle \left|i \pm \overline{i+1}\right\rangle |2m\rangle$,
$i = 1, 3, 5, ..., (2l-1)$.

$|\phi_{2k+2m+2l+i-1,2k+2m+2l+i}\rangle = \left|i \pm \overline{i+1}\right\rangle |0\rangle |m\rangle$,
$i = 2, 4, 6, ..., (2k-2)$.

$|\phi_{4k+2m+2l-1,4k+2m+2l}\rangle = |0 \pm 1\rangle |0\rangle |m\rangle$

$|\phi_{4k+2m+2l+i-1,4k+2m+2l+i}\rangle = |k\rangle \left|i \pm \overline{i+1}\right\rangle |0\rangle$,
$i = 2, 4, 6, ..., (2l-2)$.

$|\phi_{4k+4l+2m-1,4k+4l+2m}\rangle = |k\rangle |0 \pm 1\rangle |0\rangle$

$|\phi_{4k+4l+2m+i-1,4k+4l+2m+i}\rangle = |0\rangle |l\rangle \left|i \pm \overline{i+1}\right\rangle,$
$i = 2, 4, 6, ..., (2m-2).$
$|\phi_{4k+4l+4m-1,4k+4l+4m}\rangle = |0\rangle |l\rangle |0 \pm 1\rangle$
$|\phi_{4k+4l+4m+i}\rangle = |i\rangle |l\rangle |m\rangle, \ i = 1, 2, 3, ...(2k-1).$
$|\phi_{6k+4l+4m+i-1}\rangle = |k\rangle |i\rangle |m\rangle, \ i = 1, 2, 3, ...(l-1).$
$|\phi_{6k+4l+4m+i-2}\rangle = |k\rangle |i\rangle |m\rangle,$
$i = (l+1), (l+2), (l+3), ...(2l-1).$
$|\phi_{6k+6l+4m+i-3}\rangle = |k\rangle |l\rangle |i\rangle, \ i = 1, 2, 3, ...(m-1).$
$|\phi_{6k+6l+4m+i-4}\rangle = |k\rangle |l\rangle |i\rangle,$
$i = (m+1), (m+2), (m+3), ...(2m-1).$
cannot be perfectly distinguished by LOCC.

Theorem 4: In $\mathbb{C}^{2k+1}\bigotimes\mathbb{C}^{2l+1}\bigotimes\mathbb{C}^{2m}$, the set of $6(k+l+m)-8$ orthogonal product states

$|\phi_{i,i+1}\rangle=\left|i\pm\overline{i+1}\right\rangle|2l\rangle\,\overline{|m-1\rangle}$, $i=1,3,5,...,(2k-1)$.

$|\phi_{2k+1+i,2k+2+i}\rangle=\left|i\pm\overline{i+1}\right\rangle|0\rangle|m-1\rangle$, $i=0,2,4,...,(2k-2)$.

$|\phi_{4k+i,4k+1+i}\rangle=|2\rangle\left|i\pm\overline{i+1}\right\rangle|2m-1\rangle$, $i=1,3,5,...,(2l-1)$.

$|\phi_{4k+2l+1+i,4k+2l+2+i}\rangle=|2\rangle\left|i\pm\overline{i+1}\right\rangle|0\rangle$, $i=0,2,4,...,(2l-2)$.

$|\phi_{4k+4l+1+i,4k+4l+2+i}\rangle=|2k\rangle|1\rangle\left|i\pm\overline{i+1}\right\rangle$,
$i=0,2,4,...,(2m-4)$.

$|\phi_{4k+4l+2m-2+i,4k+4l+2m-1+i}\rangle=|0\rangle|1\rangle\left|i\pm\overline{i+1}\right\rangle$,
$i=1,3,5,...,(2m-3)$.

$|\phi_{4k+4l+4m-3,4k+4l+4m-2}\rangle=|1\rangle|1\rangle\left|\overline{2m-2}\pm\overline{2m-1}\right\rangle$

$|\phi_{4k+4l+4m-2+i}\rangle=|i\rangle|1\rangle|m-1\rangle$, $i=1,2,3,...,(2k-1)$.

$|\phi_{6k+4l+4m-4+i}\rangle=|2\rangle|i\rangle|m-1\rangle$, $i=2,3,4,...,(2l-1)$.

$|\phi_{6k+6l+4m-5+i}\rangle=|2\rangle|1\rangle|i\rangle$, $i=1,2,3,...,(m-2)$.

Theorem 5: In $\mathbb{C}^{2k+1} \bigotimes \mathbb{C}^{2l} \bigotimes \mathbb{C}^{2m}$, the set of $6(k+l+m)-11$ orthogonal product states

$|\phi_{i,i+1}\rangle = |i \pm \overline{i+1}\rangle |2l-1\rangle |m-1\rangle$, $i = 1, 3, 5, ..., (2k-1)$.
$|\phi_{2k+1+i,2k+2+i}\rangle = |i \pm \overline{i+1}\rangle |0\rangle |m-1\rangle$, $i = 0, 2, 4, ..., (2k-2)$.
$|\phi_{4k+i,4k+1+i}\rangle = |2\rangle |i \pm \overline{i+1}\rangle |2m-1\rangle$, $i = 1, 3, 5, ..., (2l-3)$.
$|\phi_{4k+2l-1+i,4k+2l+i}\rangle = |2\rangle |i \pm \overline{i+1}\rangle |0\rangle$, $i = 0, 2, 4, ..., (2l-4)$.
$|\phi_{4k+4l-3,4k+4l-2}\rangle = |2\rangle |\overline{2l-2} \pm \overline{2l-1}\rangle |2m-2\rangle$
$|\phi_{4k+4l-1+i,4k+4l+i}\rangle = |2k\rangle |1\rangle |i \pm \overline{i+1}\rangle$, $i = 0, 2, 4, ..., (2m-4)$.
$|\phi_{4k+4l+2m-4+i,4k+4l+2m-3+i}\rangle = |0\rangle |1\rangle |i \pm \overline{i+1}\rangle$,
$i = 1, 3, 5, ..., (2m-3)$.
$|\phi_{4k+4l+4m-5,4k+4l+4m-4}\rangle = |1\rangle |1\rangle |\overline{2m-2} \pm \overline{2m-1}\rangle$
$|\phi_{4k+4l+4m-4+i}\rangle = |i\rangle |1\rangle |m-1\rangle$, $i = 1, 2, 3, ..., (2k-1)$.
$|\phi_{6k+4l+4m-6+i}\rangle = |2\rangle |i\rangle |m-1\rangle$, $i = 2, 3, 4, ..., (2l-2)$.
$|\phi_{6k+6l+4m-8+i}\rangle = |2\rangle |1\rangle |i\rangle$, $i = 1, 2, 3, ..., (m-2)$.
$|\phi_{6k+6l+4m-9+i}\rangle = |2\rangle |1\rangle |i\rangle$, $i = m, (m+1), (m+2)..., (2m-2)$.

cannot be perfectly distinguished by LOCC.

Theorem 6: In $\mathbb{C}^{2k}\bigotimes\mathbb{C}^{2l}\bigotimes\mathbb{C}^{2m}$, the set of $6(k+l+m)-14$ orthogonal product states

$|\phi_{i+1,i+2}\rangle=\left|i\pm\overline{i+1}\right\rangle\left|\overline{2l-1}\right\rangle|2\rangle$, $i=0,2,4,...,(2k-4)$.

$|\phi_{2k-2+i,2k-1+i}\rangle=\left|i\pm\overline{i+1}\right\rangle|0\rangle|2\rangle$, $i=1,3,5,...,(2k-5)$.

$|\phi_{4k-5,4k-4}\rangle=\left|\overline{2k-2}\pm\overline{2k-1}\right\rangle|0\rangle|2\rangle$

$|\phi_{4k-3+i,4k-2+i}\rangle=|2\rangle\left|i\pm\overline{i+1}\right\rangle\left|\overline{2m-1}\right\rangle$, $i=0,2,4,...,(2l-4)$.

$|\phi_{4k+2l-6+i,4k+2l-5+i}\rangle=|2\rangle\left|i\pm\overline{i+1}\right\rangle|0\rangle$, $i=1,3,5,...,(2l-5)$.

$|\phi_{4k+4l-9,4k+4l-8}\rangle=|2\rangle\left|\overline{2l-2}\pm\overline{2l-1}\right\rangle|0\rangle$

$|\phi_{4k+4l-7+i,4k+4l-6+i}\rangle=|2k-1\rangle|2\rangle\left|i\pm\overline{i+1}\right\rangle$,
$i=0,2,4,...,(2m-4)$.

$|\phi_{4k+4l+2m-10+i,4k+4l+2m-9+i}\rangle=|0\rangle|2\rangle\left|i\pm\overline{i+1}\right\rangle$,
$i=1,3,5,...,(2m-5)$.

$$\left|\phi_{4k+4l+4m-13,4k+4l+4m-12}\right\rangle = |0\rangle\,|2\rangle\,\left|\overline{2m-2}\pm\overline{2m-1}\right\rangle$$
$$\left|\phi_{4k+4l+4m-11,4k+4l+4m-10}\right\rangle = \left|\overline{2k-3}\pm\overline{2k-2}\right\rangle|1\rangle\,|2\rangle$$
$$\left|\phi_{4k+4l+4m-9,4k+4l+4m-8}\right\rangle = |2\rangle\left|\overline{2l-3}\pm\overline{2l-2}\right\rangle|1\rangle$$
$$\left|\phi_{4k+4l+4m-7,4k+4l+4m-6}\right\rangle = |1\rangle\,|2\rangle\left|\overline{2m-3}\pm\overline{2m-2}\right\rangle$$
$$\left|\phi_{4k+4l+4m-7+i}\right\rangle = |2\rangle\,|2\rangle\,|i\rangle,\ i = 2,3,...,(2m-2).$$
$$\left|\phi_{4k+4l+6m-11+i}\right\rangle = |i\rangle\,|2\rangle\,|2\rangle,\ i = 3,4,...,(2k-2).$$
$$\left|\phi_{6k+4l+6m-15+i}\right\rangle = |2\rangle\,|i\rangle\,|2\rangle,\ i = 3,4,...,(2l-2).$$
$$\left|\phi_{6k+6l+6m-16}\right\rangle = |2\rangle\,|2\rangle\,|1\rangle$$
$$\left|\phi_{6k+6l+6m-15}\right\rangle = |1\rangle\,|2\rangle\,|2\rangle$$
$$\left|\phi_{6k+6l+6m-14}\right\rangle = |2\rangle\,|1\rangle\,|2\rangle$$

cannot be perfectly distinguished by LOCC.

Contd..

- One may observe that each class of product states constructed above are not an UPB. The sets can be extended to orthonormal product bases. All the sets of states cannot be perfectly distinguished by LOCC.

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- This is because of the fact that for local distinguishability it is necessary to eliminate state(s) of a given set, which is not possible in the given scenario.
- The special about the sets(Th.1-Th.6) is that they are the LOCC indistinguishable sets containing minimum number of states with respect to their dimensions.

Contd..

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- This is because of the fact that for local distinguishability it is necessary to eliminate state(s) of a given set, which is not possible in the given scenario.
- The special about the sets(Th.1-Th.6) is that they are the LOCC indistinguishable sets containing minimum number of states with respect to their dimensions.
- Because of the symmetric structure of those sets it indicates that no one of the three parties cannot eliminate any state from those sets.

Entanglement assisted Discrimination protocol

Example 3: In $\mathbb{C}^6 \bigotimes \mathbb{C}^6 \bigotimes \mathbb{C}^6$ the set of 36 orthogonal product states

$$
\left\{
\begin{array}{cc}
|\psi_1\rangle = |2+3\rangle\,|0\rangle\,|3\rangle & |\psi_4\rangle = |2+3\rangle\,|2\rangle\,|5\rangle \\
|\psi_2\rangle = |3\rangle\,|2+3\rangle\,|0\rangle & |\psi_5\rangle = |5\rangle\,|2+3\rangle\,|2\rangle \\
|\psi_3\rangle = |0\rangle\,|3\rangle\,|2+3\rangle & |\psi_6\rangle = |2\rangle\,|5\rangle\,|2+3\rangle \\
|\psi_{7,8}\rangle = |2\pm 3\rangle\,|2\rangle\,|3\rangle & |\psi_{13,14}\rangle = |3\pm 4\rangle\,|2\rangle\,|4\rangle \\
|\psi_{9,10}\rangle = |3\rangle\,|2\pm 3\rangle\,|2\rangle & |\psi_{15,16}\rangle = |4\rangle\,|3\pm 4\rangle\,|2\rangle \\
|\psi_{11,12}\rangle = |2\rangle\,|3\rangle\,|2\pm 3\rangle & |\psi_{17,18}\rangle = |2\rangle\,|4\rangle\,|3\pm 4\rangle \\
|\psi_{19,20}\rangle = |1\pm 2\rangle\,|1\rangle\,|3\rangle & |\psi_{25,26}\rangle = |0\pm 1\rangle\,|0\rangle\,|3\rangle \\
|\psi_{21,22}\rangle = |3\rangle\,|1\pm 2\rangle\,|1\rangle & |\psi_{27,28}\rangle = |3\rangle\,|0\pm 1\rangle\,|0\rangle \\
|\psi_{23,24}\rangle = |1\rangle\,|3\rangle\,|1\pm 2\rangle & |\psi_{29,30}\rangle = |0\rangle\,|3\rangle\,|0\pm 1\rangle \\
|\psi_{31,32}\rangle = |4\pm 5\rangle\,|2\rangle\,|5\rangle & \\
|\psi_{33,34}\rangle = |5\rangle\,|4\pm 5\rangle\,|2\rangle & \\
|\psi_{35,36}\rangle = |2\rangle\,|5\rangle\,|4\pm 5\rangle &
\end{array}
\right\}
$$

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Contd..

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- Since, all the sets are symmetric, it is really not important which pair of parties holds the resource state. Also the required entanglement resource to accomplish the task of distinguishing the non-local sets does not depend on the dimension of the subsystems.
- All the LOCC indistinguishable sets constructed in Th.1-Th.6 are minimal in cardinality. Also, the above classes of states are minimal sets which are distinguishable through GHZ state. We observe advantage by using genuine entanglement as a resource for discrimination of some of those classes.

Contd..

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- Second, the question of optimality of the entangled resources used in our discrimination protocols remains open.
- In a given Hilbert space, the subsets constructed here are small sets. That is the number of states contained in the set is much lesser than the net dimension of the Hilbert space. Hence an essential search in this direction is to find out the number of orthogonal product states from which no one can be eliminated by orthogonality preserving positive-operator-valued measurement(OPPOVM). Explicit constructions of such sets are also important for any multipartite quantum system.

Activation of nonlocality[Phys. Rev. A 110, 042424 (2024)]

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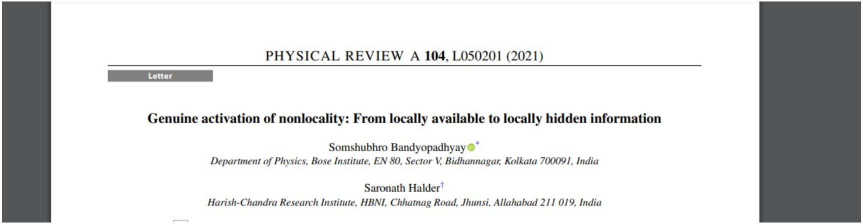
- Here comes some trivial sets with this property and that initiated the perception of local redundancy. An orthogonal set is pronounced to be locally redundant if it remains orthogonal after discarding one or more subsystems. Apart from that, it is called to be locally irredundant.



PHYSICAL REVIEW A **104**, L050201 (2021)

Letter

Genuine activation of nonlocality: From locally available to locally hidden information

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Activation of nonlocality

- In [Phys. Rev. A 104, L050201 (2021)], Bandyopadhyay and Halder first revealed that there exist some orthogonal sets free from local redundancy that can be distinguished locally but under some orthogonality preserving local measurement(OPLM), each outcome will lead to a locally indistinguishable set. They called this phenomenon genuine activation of nonlocality. The term 'genuine' denotes that the set must be free from local redundancy.

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- In [New J. Phys. 24 043036 (2022)], Li and Zheng extended the result by addressing two types of genuine hidden nonlocality (Genuine hidden nonlocality of type I or II) and came up with a locally distinguishable set of bipartite product states which can be deterministically converted to a locally irreducible set via OPLM.

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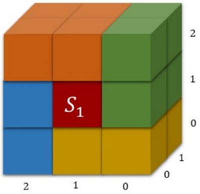
- After that Ghosh et al. [Phys. Rev. A 106, L010202, 2022] gave a scenario for the activation of the strongest possible

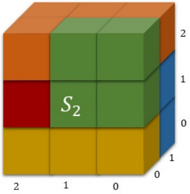
- The genuine activation of nonlocality for the multipartite scenario is not fully explored. Still, many questions arise which are not been fully known yet.
- Is there exist a local set in which local activation is not possible, but to activate nonlocality two or more parties must come to the same place?

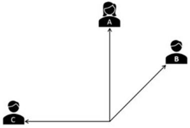
- The genuine activation of nonlocality for the multipartite scenario is not fully explored. Still, many questions arise which are not been fully known yet.
- Is there exist a local set in which local activation is not possible, but to activate nonlocality two or more parties must come to the same place?
- From the point of view of hidden nonlocality activation, these types of sets belong to the more local category than aforesaid sets.
- In our work, we have tried to explore the solution to the above question from the perspective of LPCC (P stands for projective measurments).
- More precisely, we encounter two distinct types of local sets within the n-party quantum system, say S_1 and S_2. None of these sets can be directly employed to transmit hidden classical information because we know the information encoded in a locally distinguishable set is always locally accessible.

Examples: $S_1, S_2 \subset \mathbb{C}^{3\otimes 2\otimes 3}$

- But a possibility arises when we consider the potentiality to activate nonlocality within S_1 through local operations. In the context of data hiding or secret sharing it is possible to effectively turn such a resource-less set into a resource-rich one. We show that such a set can be converted to a locally indistinguishable orthogonal set by LOCC with certainty. This transformation allows us to locally hide information within S_1. i.e., the information encoded in S_1 which was locally available initially, after transmission, can not be accessed completely by the local observers, part of it will always remain hidden.





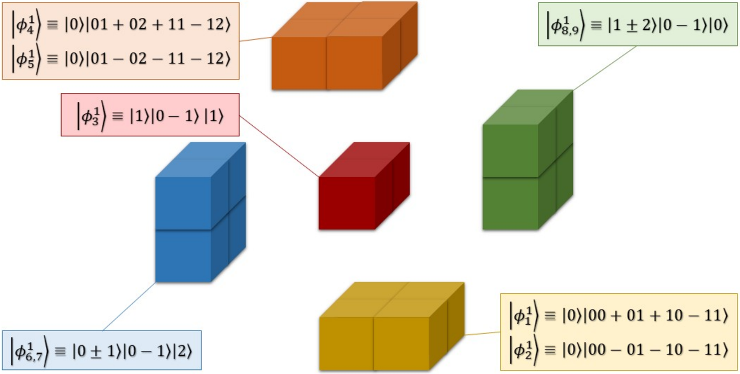


$S_1, S_2 \subset \mathcal{H}_{ABC} \equiv \mathbb{C}^{3\otimes 2\otimes 3}$

Contains nine mutually orthogonal pure states

Locally distinguishable under LPCC

Example 1: $S_1 \subset \mathbb{C}^{3\otimes 2\otimes 3}$



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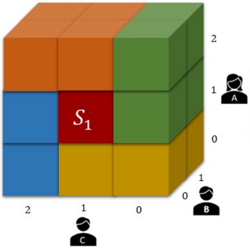

Locally irredundant

Nonlocality can be activated by LPCC

Example 2: $S_2 \subset \mathbb{C}^{3\otimes 2\otimes 3}$

- On the other hand, S_2 represents such sets where genuine activation by LOCC is impossible. So, the classical information encoded with the orthogonal states of such set always remains accessible by the local observers, i.e., when parties are constrained to perform only OPLM, the outcomes consistently exhibit locality. This characteristic places S_2 on the more local end of the spectrum compared to S_1 as S_2 is seemingly unsuitable for applications like secret sharing and data hiding.

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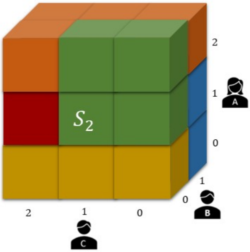
- Now this prompts a compelling inquiry: Could S_2 still contain hidden nonlocal characteristics that are yet to be unveiled? If so, the nature of this hidden nonlocality becomes a central question, along with the pursuit of methods to access it.

Example 2: $S_2 \subset \mathbb{C}^{3\otimes2\otimes3}$

- For that regard, we give an example of a local set whose nonlocality cannot retained by local observers but two or more observers can jointly grab its hidden nonlocality.

Example 2: $S_2 \subset \mathbb{C}^{3\otimes 2\otimes 3}$



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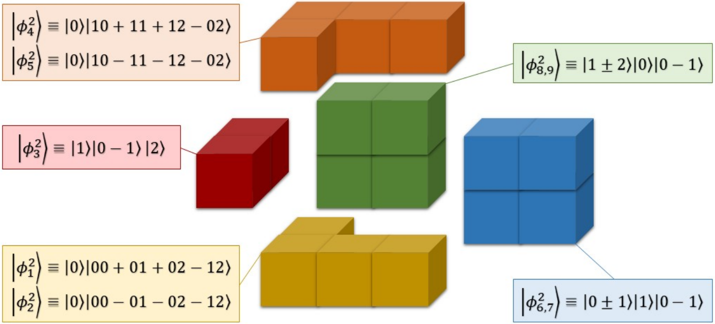

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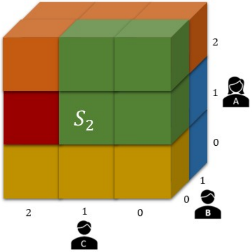
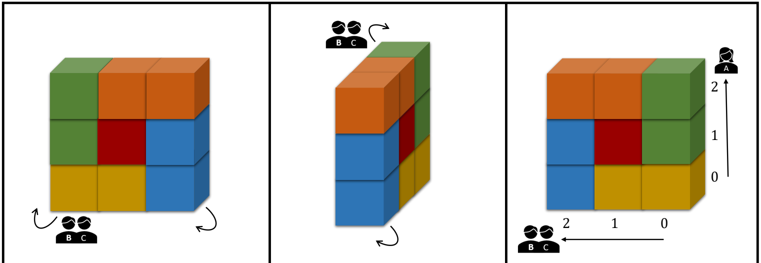


Locally irredundant

Nonlocality cannot be activated by LPCC

Bob and Charlie jointly can activate nonlocality by giving PVMs

Example 2: $S_2 \subset \mathbb{C}^{3\otimes 2\otimes 3}$



(a) (b) (c)

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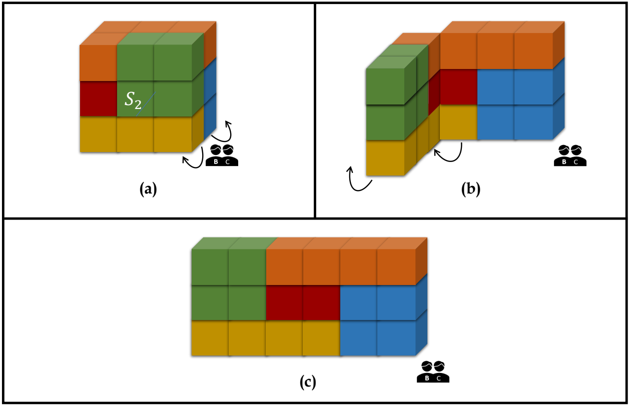

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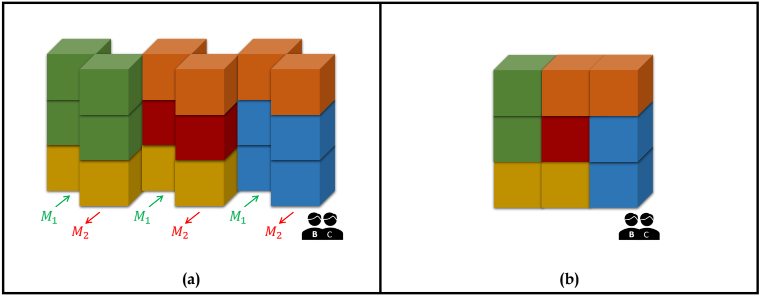

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- We also find some asymmetric configurations in tripartite systems which provide the phenomenon of nonlocality without entanglement, Quantum Inf. Process. 21, 169 (2022).
- It is also worthful to discuss with round number required to achieve local discrimination tasks, J. Phys. A: Math. Theor. 56 (2023) 365303.
- Also there are some strong form of quantum nonlocality exist, if we consider Unextendible biseparable basis beyond unextendible product basis, Phys. Rev. A. 109, 052211 (2024).

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