Quantum Incompatibility in Parallel vs Antiparallel Spins

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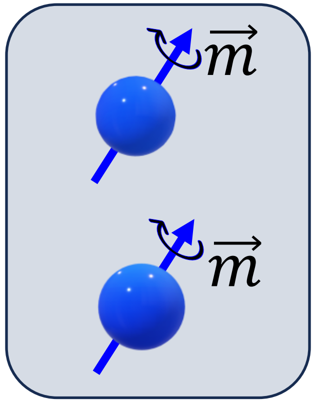

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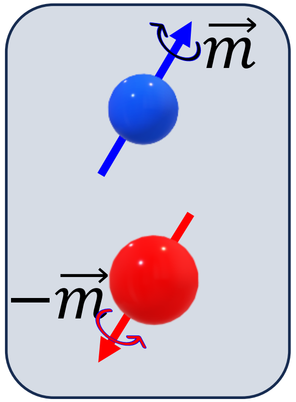

Manik Banik

S. N. Bose National Centre for Basic Sciences

Quantum Incompatibility in Parallel vs Antiparallel Spins

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[2]*Department Of Mathematics, Balagarh Bijoy Krishna Mahavidyalaya, Balagarh, Hooghly-712501, West Bengal, India.*

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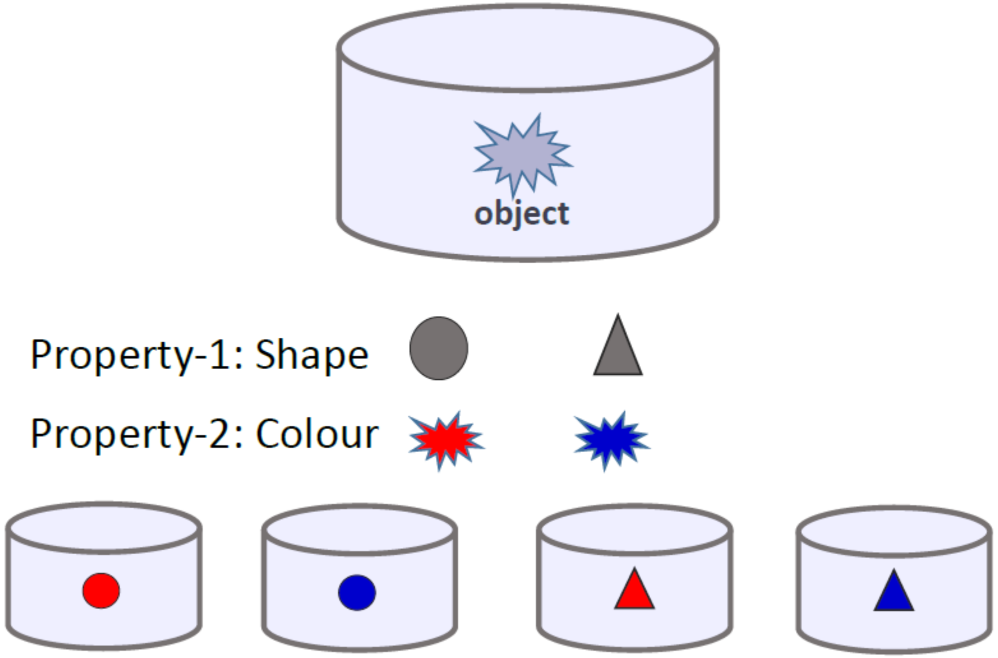
We investigate the joint measurability of incompatible quantum observables on ensembles of parallel and antiparallel spin pairs. In the parallel configuration, two systems are identically prepared, whereas the antiparallel configuration pairs a system with its spin-flipped counterpart. We demonstrate that the antiparallel configuration enables the exact simultaneous prediction of three mutually orthogonal spin components—an advantage unattainable in the parallel case. As we show, this enhanced measurement compatibility in the antiparallel configuration is better explained within the framework of generalized probabilistic theories, which allow a broader class of composite structures while preserving quantum descriptions at the subsystem level. Furthermore, this approach extends the study of measurement incompatibility to more general configurations beyond just the parallel and antiparallel cases, providing deeper insight into the boundary between physical and unphysical quantum state evolutions.

Classical World



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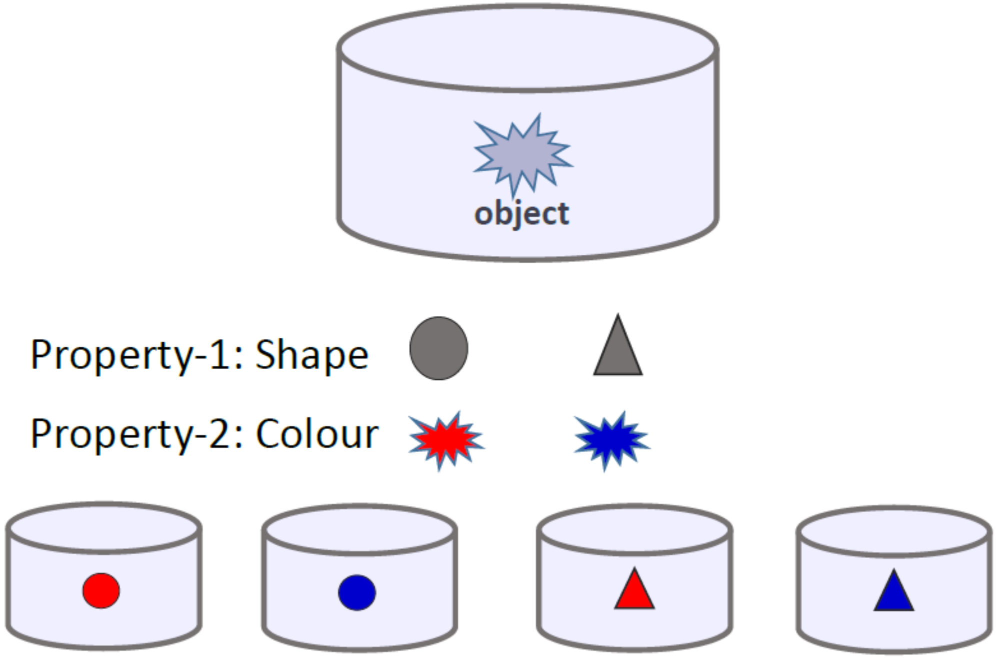

Quantum World

- In Quantum world, however, the complementarity principle holds. There are certain pairs of complementary properties that cannot all be observed or measured simultaneously.
- First, pointed out by **Niels Bohr's** in **1927** at the **Como Conference** in Italy.

- N. Bohr, *The Quantum Postulate and the Recent Development of Atomic Theory*, Nature **121**, 580–590 (1928).
- De Gregorio, *Bohr's way to defining complementarity*, Stud. Hist. Philos. Sci. B **45**, 72–82 (2014).

Complementarity: Example

- ✓ Path information and interference visibility in the double-slit experiment

REVIEW ARTICLE

Quantum optical tests of complementarity

Marlan O. Scully, Berthold-Georg Englert & Herbert Walther

Simultaneous observation of wave and particle behaviour is prohibited, usually by the position–momentum uncertainty relation. New detectors, constructed with the aid of modern quantum optics, provide a way around this obstacle in atom interferometers, and allow the investigation of other mechanisms that enforce complementarity.

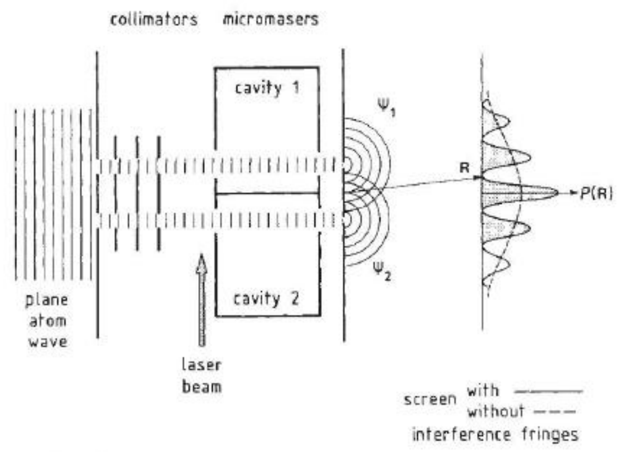

FIG. 3 Two-slit experiment with atoms. A set of wider slits collimates two atom beams which illuminate the narrow slits where the interference pattern originates. The collimation of the atomic beams would actually be done using atomic optics. One could, for instance, employ six-pole fields operating either on the magnetic dipole moment, or in the case of Rydberg atoms on the field-induced electric dipole moment. This set-up is supplemented by two high-quality micromaser cavities and a laser beam to to provide which-path information.

- ✓ Non-commuting observables such as position and momentum, or spin components along different axes [1-3]

---

1. E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, 1976)
2. P. J. Lahti, *Uncertainty and complementarity in axiomatic quantum mechanics*, Int. J. Theo. Phys. **19**, 789–842 (1980)
3. P. Busch, *Indeterminacy relations and simultaneous measurements in quantum theory*, Int. J. Theo. Phys. **24**, 63–92 (1985)

Joint measurement of incompatible observables

✓ The development of generalized measurements, formalized via positive operator-valued measures (POVMs), demonstrates that incompatible observables can, in fact, be jointly measured – albeit with an inherent degree of fuzziness or imprecision [1-2]

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- ✓ Spin-1/2 system $\rho_{\vec{m}} = \frac{1}{2}(\mathbf{1} + \vec{m}\cdot\vec{\sigma}) \in \mathcal{D}(\mathbb{C}^2)$ $\sigma_{\hat{n}} \equiv \{P^a_{\hat{n}} = \frac{1}{2}(\mathbf{1} + a\,\hat{n}\cdot\vec{\sigma})\}$

$$\sigma_{\hat{n},\lambda} \equiv \{P^a_{\hat{n},\lambda} = \frac{1}{2}(\mathbf{1} + \lambda\, a\, \hat{n}\cdot\vec{\sigma})\}$$

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Definition 1 (Busch et al. [11]). *A set $\mathcal{S}_N := \{\sigma_{\hat{n}_j,\lambda}\}_{j=1}^N$ of N unsharp spin observables is jointly measurable if there exists a POVM $\mathcal{G} \equiv \{\pi_{\vec{a}} \geq 0 \,|\, \sum_{\vec{a}} \pi_{\vec{a}} = \mathbf{1}\}$, with outcome strings $\vec{a} = [a_1, \ldots, a_N]$, such that each observable appears as a marginal, i.e. $P^{a_j}_{\hat{n}_j,\lambda} = \sum_{\vec{a}\setminus a_j} \pi_{\vec{a}}$ for all j, where $\vec{a} \setminus a_j$ denotes summation over all components except a_j.*

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Spin Observables along X and Y directions are compatible up-to the sharpness value $\lambda=1/\sqrt{2}$, while observables along X,Y,Z are compatible up-to $\lambda=1/\sqrt{3}$

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Quantum incompatible: other facets

✓ **More recently, measurement incompatibility has been shown to be intimately connected to other nonclassical phenomena, such as Bell nonlocality and Einstein-Podolsky-Rosen steering [1-2].**

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- ✓ This recognition has motivated a deeper exploration of incompatibility, including scenarios involving multiple copies of a quantum system per experimental run [4].

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4. C. Carmeli et al, *Quantum Incompatibility in Collective Measurements*, Mathematics **4**, 54 (2016)

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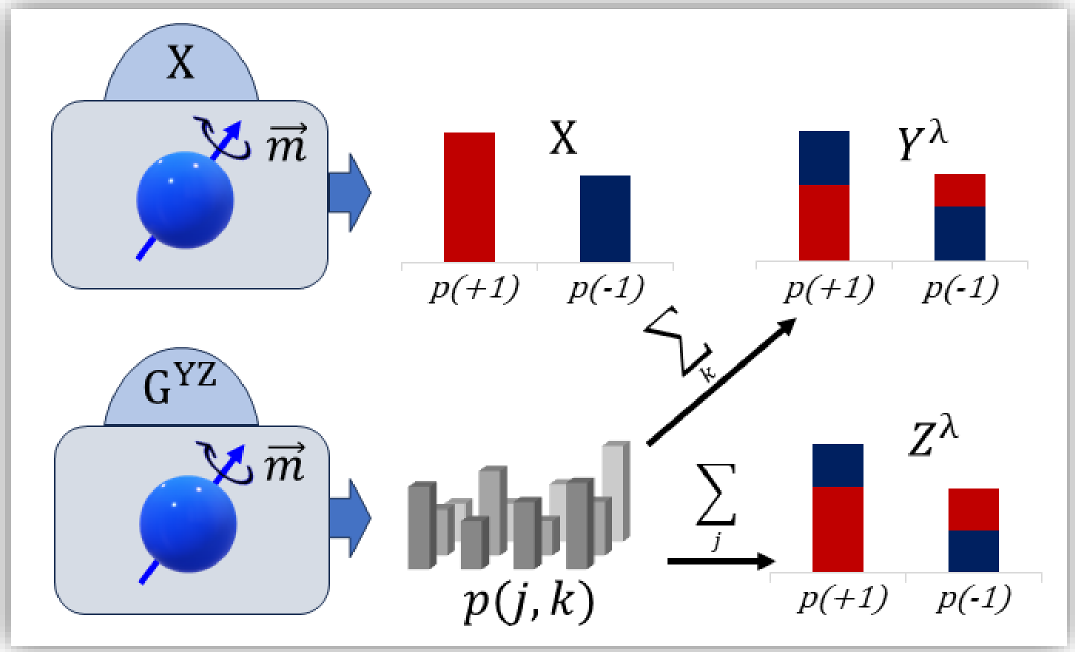

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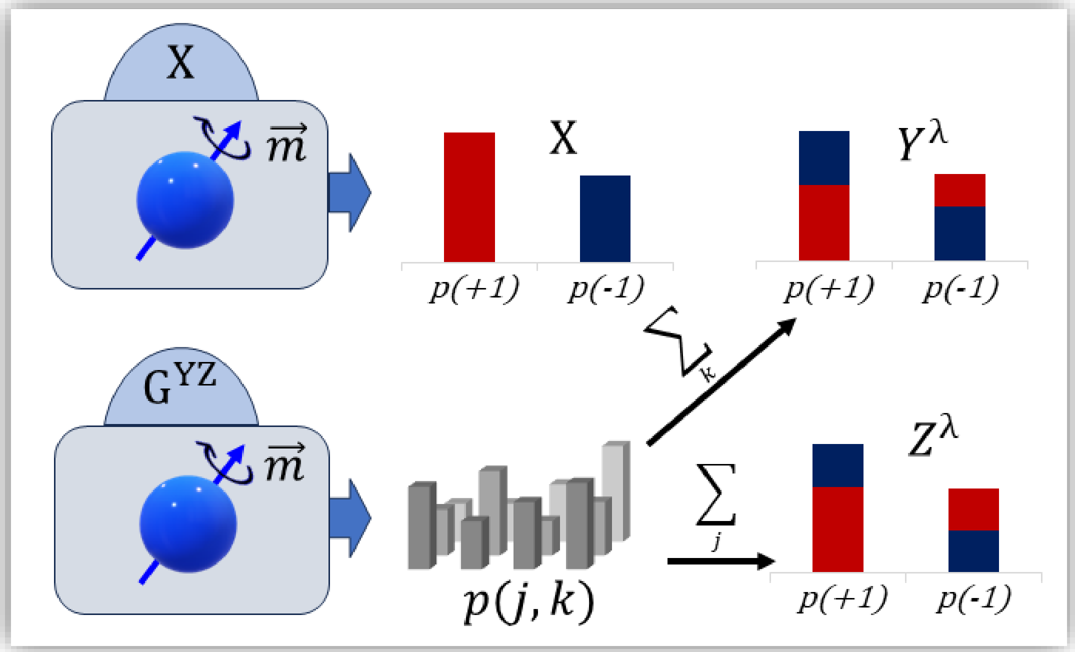
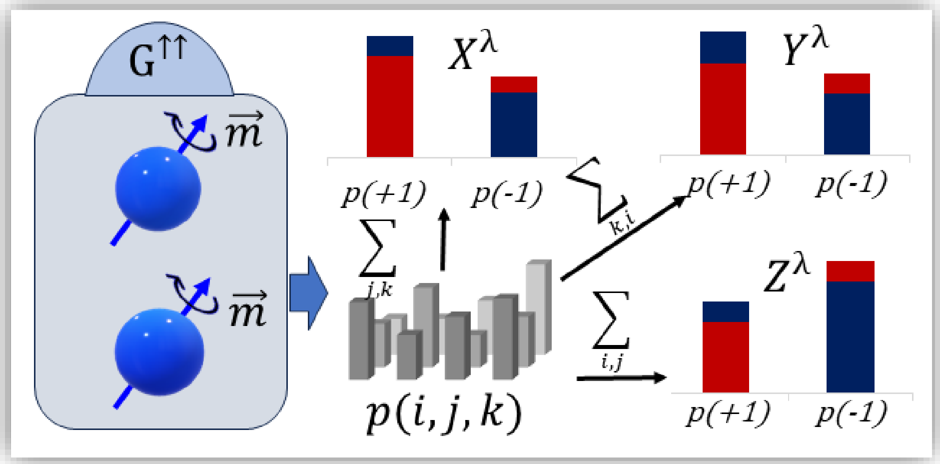


Although, X measurement become sharp, the Y and Z observable can be measured up-to the sharpness parameter value $\lambda = 1/\sqrt{2}$ [Busch'86]

Multi-copy Quantum incompatible

- ✓ The aforesaid strategy introduces an asymmetry favoring the first observable. However, Carmeli et al. showed that a more symmetric and efficient strategy is possible, one that exploits entangled across the copies while constructing the joint POVM.



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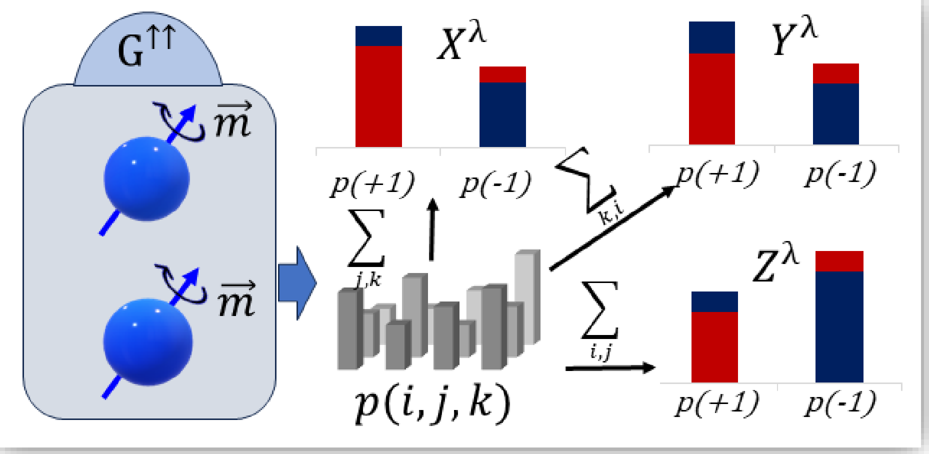

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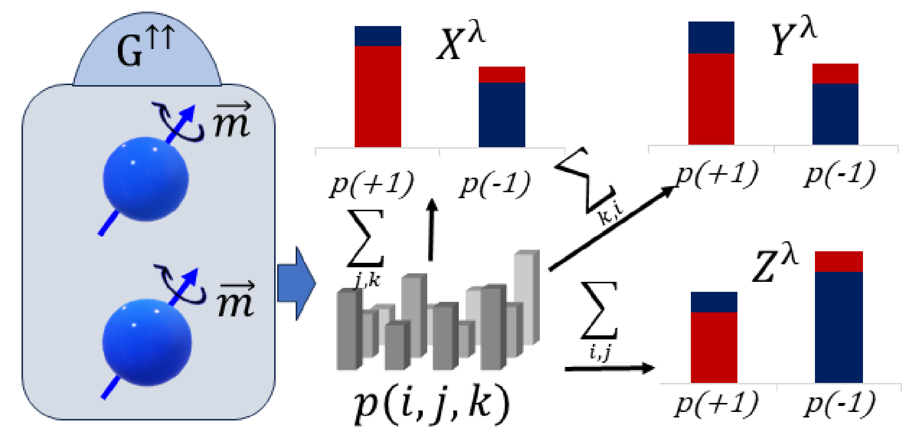

X, Y and Z observables become two-copy compatible up-to the sharpness parameter value $\lambda=\sqrt{3}/2$

$$\Pi^{\uparrow\uparrow}_{[i,j,k]} := \frac{1}{32}\Big(4\,\mathbf{1}^{\otimes 2} + \sqrt{3}\big(i\{\!\{X,\mathbf{1}\}\!\} + j\{\!\{Y,\mathbf{1}\}\!\} + k\{\!\{Z,\mathbf{1}\}\!\}\big) + ij\{\!\{X,Y\}\!\} + jk\{\!\{Y,Z\}\!\} + ki\{\!\{Z,X\}\!\}\Big), \quad (1)$$

$$\{\!\{U,V\}\!\} := U\otimes V + V\otimes U$$

Quantum incompatible: parallel vs antiparallel

- ✓ We investigate whether such enhancements persist -- or can even be improved -- when, instead of two identical (parallel) spin states, each experimental run involves one spin and its flipped counterpart (an antiparallel configuration).

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VOLUME 83, NUMBER 2 PHYSICAL REVIEW LETTERS 12 JULY 1999

Spin Flips and Quantum Information for Antiparallel Spins

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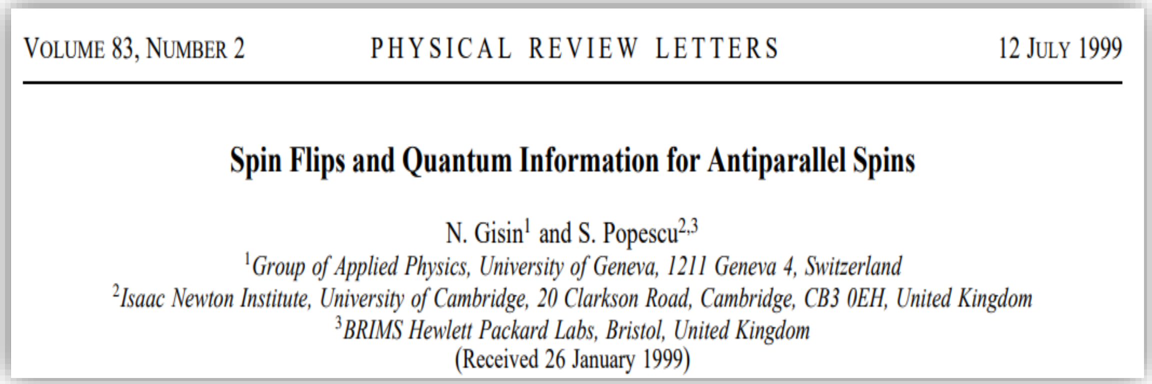
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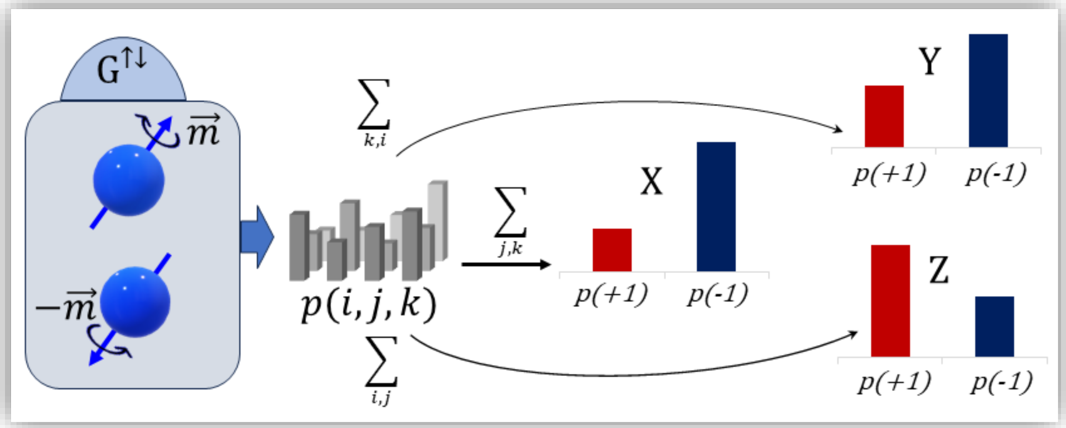

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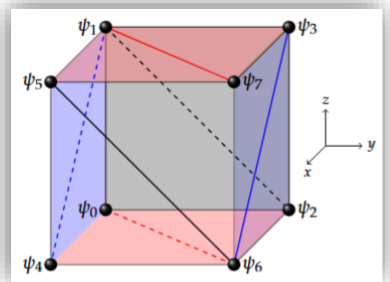

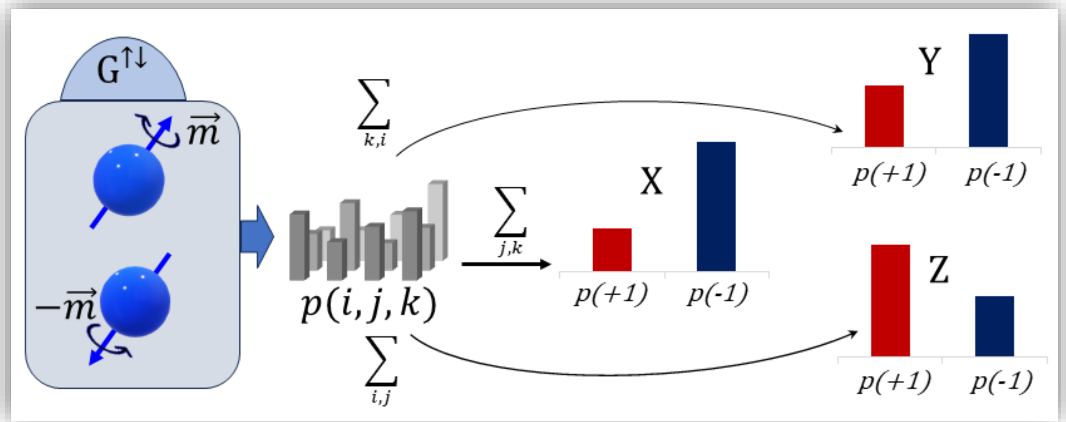

Analysis of the advantage in antiparallel case

✓ The framework of generalized probabilistic theories (GPTs) offers valuable insight.

$$\mathcal{S} \equiv (\Omega, E, T)$$

$\Omega \subset V_+$ => **a convex set embedded in some real vector space**
(un-normalized states form a convex cone)

$E \subset V_+^*$ => **dual cone**

$$e \in E \; s.t. \; e: \Omega \to [0,1]$$

$$\mathrm{M} \equiv \{e_i | e_i \in E \;\; \forall\, i, \textstyle\sum_i e_i = u\}$$

✓ I. Namioka and R. Phelps, Pac. J. Math. **31**, 469–480 (1969)
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Hilbert space $\mathcal{H}$

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$\mathcal{T}_+(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$
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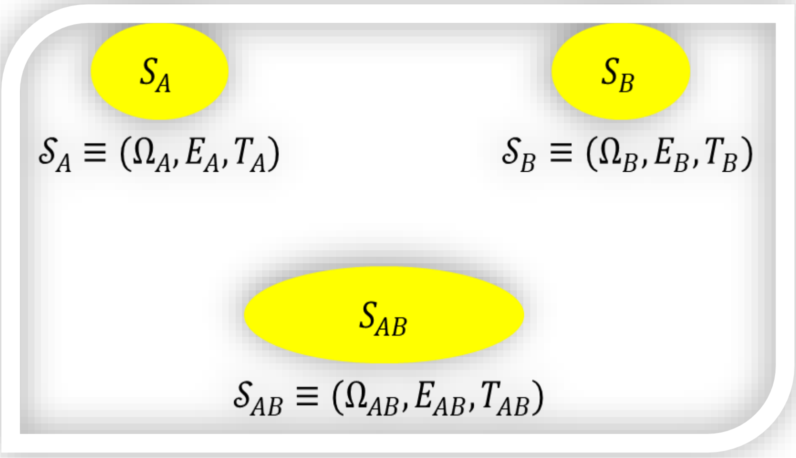

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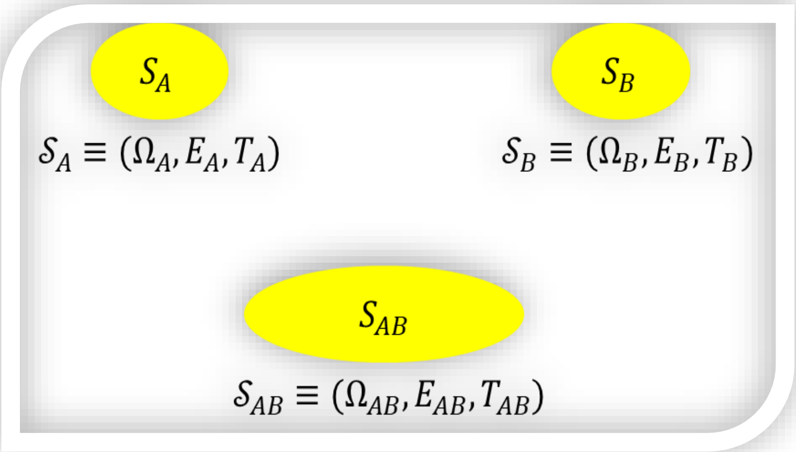

***Natural Conditions*:**

- $\forall\, \omega_{AB} \in \Omega_{AB}$ & $e_{AB} \in E_{AB}$ $\;e_{AB}(\omega_{AB}) \geq 0$
  Consistency condition
- *Joint Probabilities* $p(ab|xy)$ *must obey*
  No signaling condition
- *Marginal Statistics specifies the state uniquely*
  (Local Tomography)

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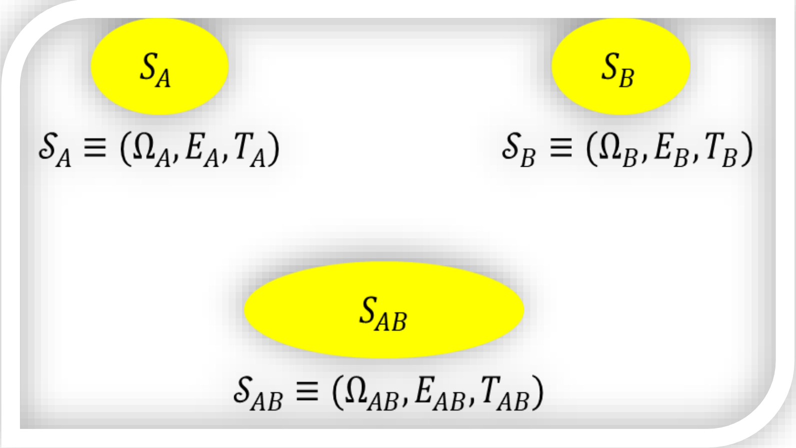
$\mathrm{M} \equiv \{e_i | e_i \in E \ \ \forall\, i, \sum_i e_i = u\}$

System
Hilbert space $\mathcal{H}$

State Cone
$\mathcal{T}_+(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$
$\mathcal{T}(\mathcal{H})$: set of all hermitian operators

Measurement
$M \equiv \{\pi_i \mid \pi_i \in \mathcal{T}_+(\mathcal{H}), \sum_i \pi_i = \mathbf{I}_{\mathcal{H}}\}$



***Natural Conditions*:**

- $\forall\ \omega_{AB} \in \Omega_{AB}\ \&\ e_{AB} \in E_{AB}\ \ e_{AB}(\omega_{AB}) \geq 0$
  Consistency condition
- *Joint Probabilities $p(ab|xy)$ must obey*
  No signaling condition
- *Marginal Statistics specifies the state uniquely*
  (Local Tomography)

Maximal Tensor Product $\boldsymbol{\Omega_A \otimes_{\{max\}} \Omega_B}$

$e_A \otimes e_B(\omega_{AB}) \geq 0$ *where* $e_A \in E_A, e_B \in E_B$

Minimal Tensor Product $\boldsymbol{\Omega_A \otimes_{\{min\}} \Omega_B}$

$e_{AB}(\omega_A \otimes \omega_B) \geq 0$ *where* $\omega_A \in \Omega_A, \omega_B \in \Omega_B$

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✓ G. Ludwig, Commun. Math. Phys.**4**, 331–348 (1967
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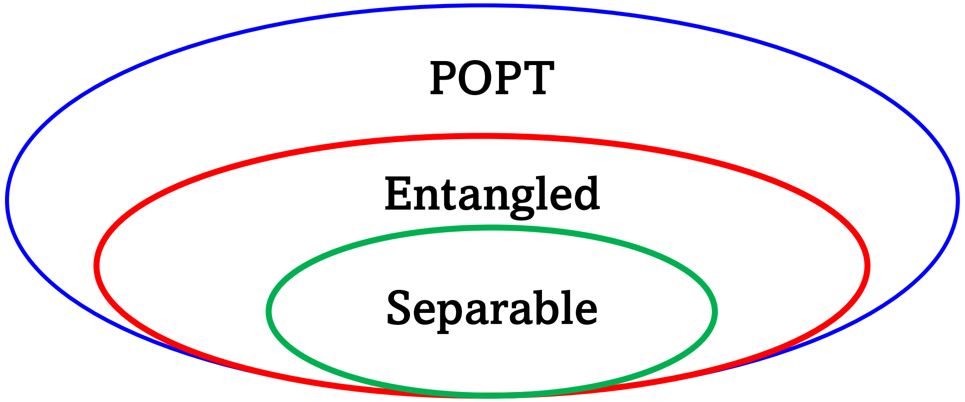
✓ G. Mackey, The Mathematical Foundations of Quantum Mechanics: A

Analysis of the advantage in antiparallel case

- ✓ The framework of generalized probabilistic theories (GPTs) offers valuable insight.

Definition 3. *In the minimal tensor product framework, the state space is given by the set of separable states:* $StateSpace = Sep(\mathbb{C}^2 \otimes \mathbb{C}^2) \subset \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. *The corresponding effect space consists of all operators* $\Pi \in \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ *satisfying* $0 \leq \mathrm{Tr}[\Pi\Omega] \leq 1$ *for all* $\Omega \in Sep(\mathbb{C}^2 \otimes \mathbb{C}^2)$.



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Analysis of the advantage in antiparallel case

Theorem 2. *The observables X^λ, Y^λ, and Z^λ are 2-copy compatible for all $\lambda \in [0,1]$ in the minimal tensor product GPT.*

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$$\Pi^{\#}_{[i,j,k]} := \frac{1}{16}\Big(2\,\mathbf{1}^{\otimes 2} + i\{\!\{X,\mathbf{1}\}\!\} + j\{\!\{Y,\mathbf{1}\}\!\} + k\{\!\{Z,\mathbf{1}\}\!\} + ij\{\!\{X,Y\}\!\} + jk\{\!\{Y,Z\}\!\} + ki\{\!\{Z,X\}\!\}\Big). \quad (3)$$

Not positive operators, but **valid effects** in minimal tensor theory.

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Generic consideration

- ✓ Two copies of the system prepared in $\rho_{\vec{m}} \otimes \Lambda(\rho_{\vec{m}})$ are available per experimental run, with Λ being a CPTP or a PTP map

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$$\mathrm{F}_\mu = \frac{1-\mu}{2}\,\mathrm{id}_2 + \frac{1+\mu}{2}\,\mathrm{F}$$

PTP for all $\mu \in [0,1]$

CPTP for $\mu \in [0,1/3]$

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This configuration offers an advantage over the parallel configuration for the joint measurability of {X, Y, Z} whenever **μ > √3 − 1**

Conclusions

- ✓ We demonstrate that the antiparallel configuration enables exact simultaneous prediction of three mutually orthogonal spin components—an advantage unattainable in the parallel case.

- ✓ As we show, this enhanced measurement compatibility in antiparallel configuration is better appreciated within the framework of generalized probabilistic theories, which allow a broader class of composite structures while preserving quantum descriptions at the subsystem level.

- ✓ Furthermore, this approach extends the study of measurement incompatibility to more general configurations beyond the parallel and antiparallel cases only, providing deeper insight into the boundary between physical and unphysical quantum state evolutions.

- ✓ At present we are extending this concept to a finite subset of states so that the reported advantage can be experimentally verified (not discussed in this talk).

"Relations between authors and referees are, of course, almost always strained. Authors are convinced that the malicious stupidity of the referee is alone preventing them from laying their discoveries before an admiring world. Referees are convinced that authors are too arrogant and obtuse to recognize blatant fallacies in their own reasoning, even when these have been called to their attention with crystalline lucidity. All physicists know this, because all physicists are both authors and referees, but it does no good. The ability of one person to hold both views is an example of what Bohr called complementarity."