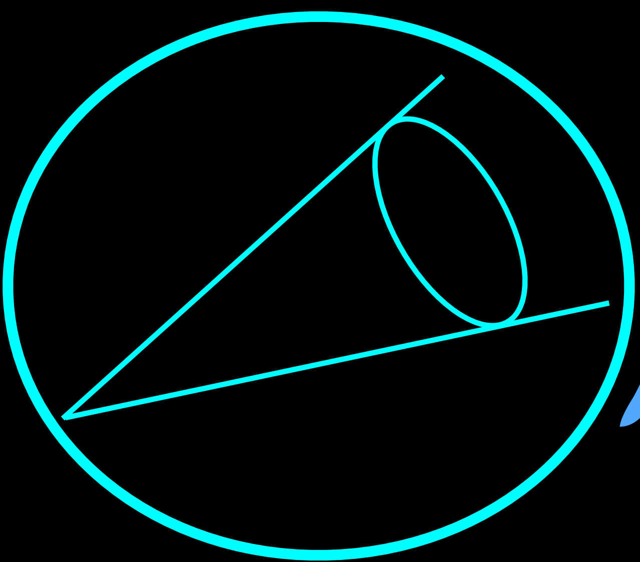
Hidden Markov Models: Classical, Quantum, and Beyond

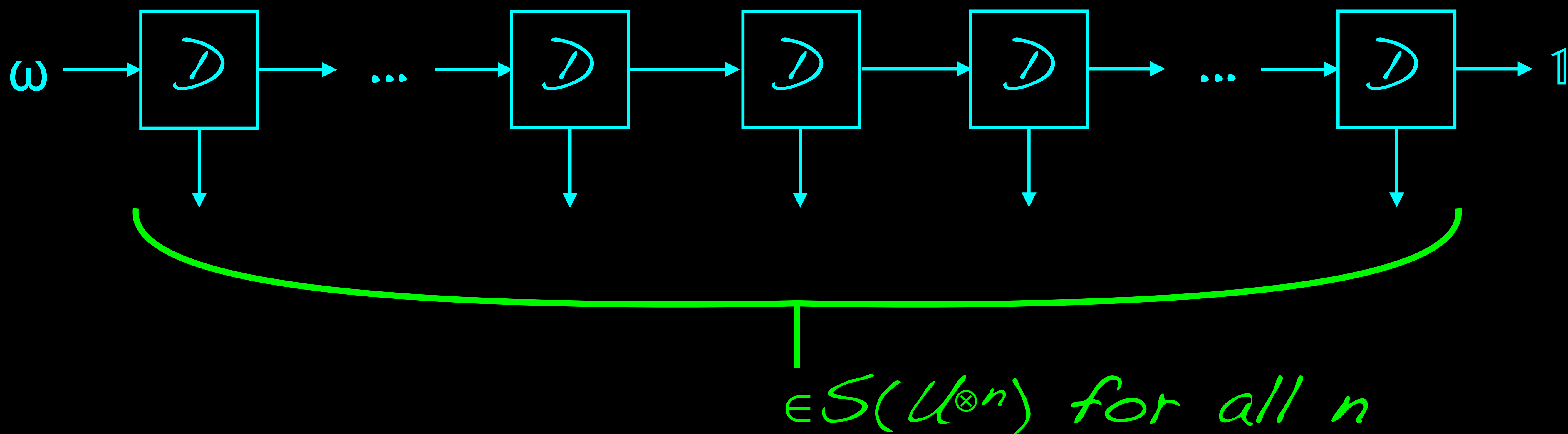


$P(uv\ldots w)$

Andreas Winter
(ICREA & UAB Barcelona)

[A. Monràs/AW, JMP 2016 – 1412.3634;
M. Fanizza/J. Lumbreras/AW, CMP 2024 – 2209.11225]

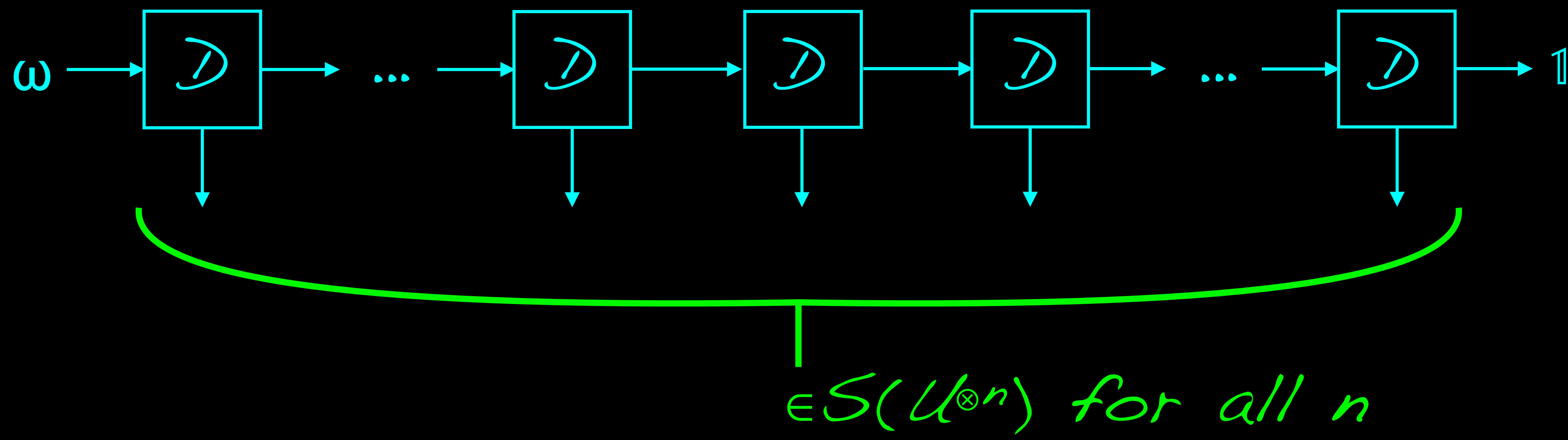
Finitely correlated state given by cptp map $\mathcal{D}: S(\mathcal{H}) \to S(\mathcal{H} \otimes \mathcal{U})$ and state $\omega = \mathrm{Tr}_{\mathcal{U}} \circ \mathcal{D}(\omega)$:



When $\mathcal{D}$ is (conjugation by) an isometry, we get matrix product state (MPS).

[Fannes/Nachtergaele/Werner, CMP 144:443-490 (1992)]

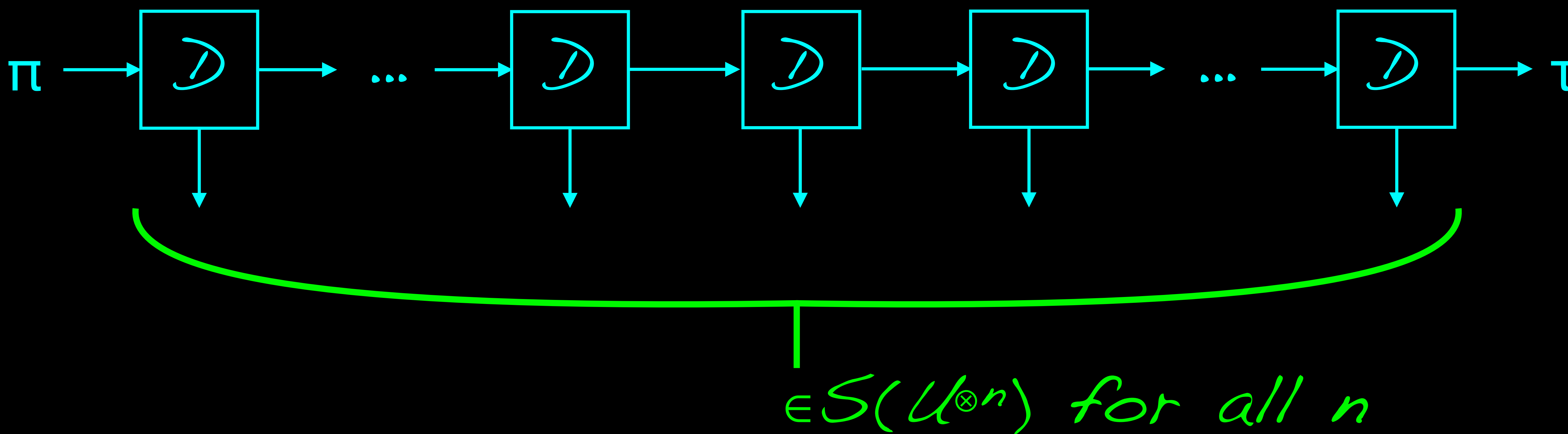
C-finitely correlated state* given by cptp map $\mathcal{D}: \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H} \otimes \mathcal{U})$ and state $\omega = Tr_{\mathcal{U}} \circ \mathcal{D}(\omega)$:



When $\mathcal{D}$ is (conjugation by) an isometry, we get matrix product state (MPS).

[Fannes/Nachtergaele/Werner, CMP 144:443-490 (1992)]

General finitely correlated state given by map $\mathcal{D}:\mathcal{V}\to\mathcal{V}\otimes\mathcal{B}(\mathcal{U})$ and $\mathcal{V}\ni\pi = \mathrm{Tr}_{\mathcal{U}}\circ\mathcal{D}(\pi)$:



$\mathcal{V}$ is a vector space, τ linear s.t. $\tau(\pi)=1$.

What does this added generality actually buy beyond C*-FCS?

[Fannes/Nachtergaele/Werner, CMP 144:443-490 (1992)]

Rest of the talk:

Classical finitely correlated states,
i.e. probability distribution on $\mathbb{U}^\infty$

Concretely, we observe an infinite time series $\dots u_{-k} \dots u_{-1} u_0 u_1 u_2 \dots u_\ell \dots$
[$u_t \in \mathbb{U}$ letters from a finite alphabet].

Rest of the talk:

Classical finitely correlated states,

i.e. probability distribution on $\mathbb{U}^\infty$

Concretely, we observe an infinite time series $\ldots u_{-k} \ldots u_{-1}\, u_0\, u_1\, u_2 \ldots u_\ell \ldots$

[$u_t \in \mathbb{U}$ letters from a finite alphabet].

Assume stationarity, i.e. for all t and ℓ,

$$\Pr\{U_1=u_1,\ldots,U_\ell=u_\ell\} = \Pr\{U_t=u_1,\ldots,U_{t+\ell-1}=u_\ell\}.$$

Rest of the talk:

Classical finitely correlated states,
i.e. probability distribution on $\mathbb{U}^\infty$

Concretely, we observe an infinite time series $\ldots u_{-k} \ldots u_{-1}\, u_0\, u_1\, u_2 \ldots u_\ell \ldots$

[$u_t \in \mathbb{U}$ letters from a finite alphabet].

Assume stationarity, i.e. for all t and ℓ,

$$\Pr\{U_1=u_1,\ldots,U_\ell=u_\ell\} = \Pr\{U_t=u_1,\ldots,U_{t+\ell-1}=u_\ell\}.$$

These marginals $P(\underline{u})$, for all finite words

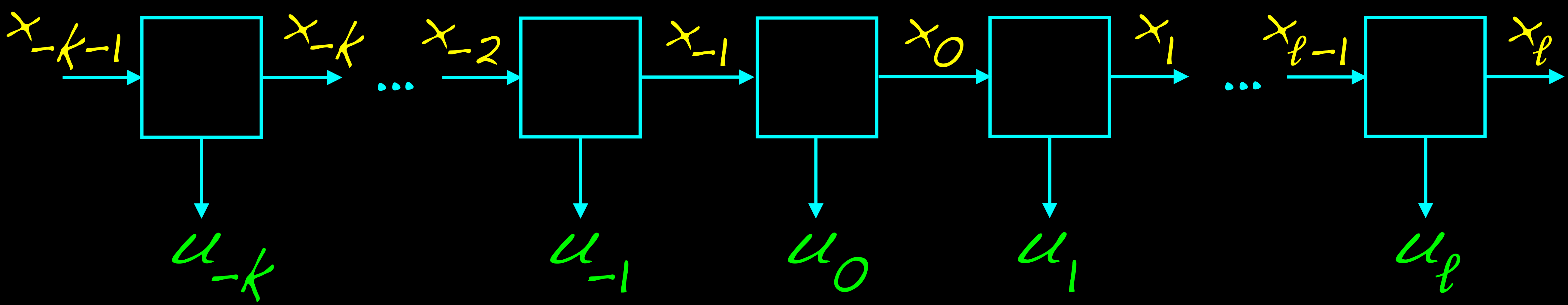
$$\underline{u} = u_1 u_2 \ldots u_\ell \in \mathbb{U}^* = \bigcup_{k\geq 0} \mathbb{U}^k,$$

determine the probability law.

Rest of the talk:

Classical finitely correlated states,
i.e. probability distribution on $\mathbb{U}^\infty$

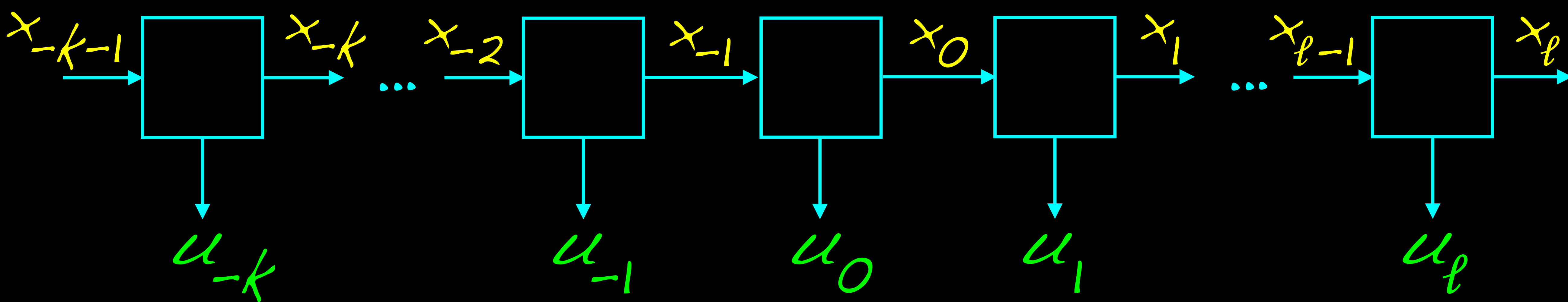
"Explanation" of $P(\underline{u})$ via a finite memory system as hidden cause:



Rest of the talk:

Classical finitely correlated states,
i.e. probability distribution on $\mathbb{U}^\infty$

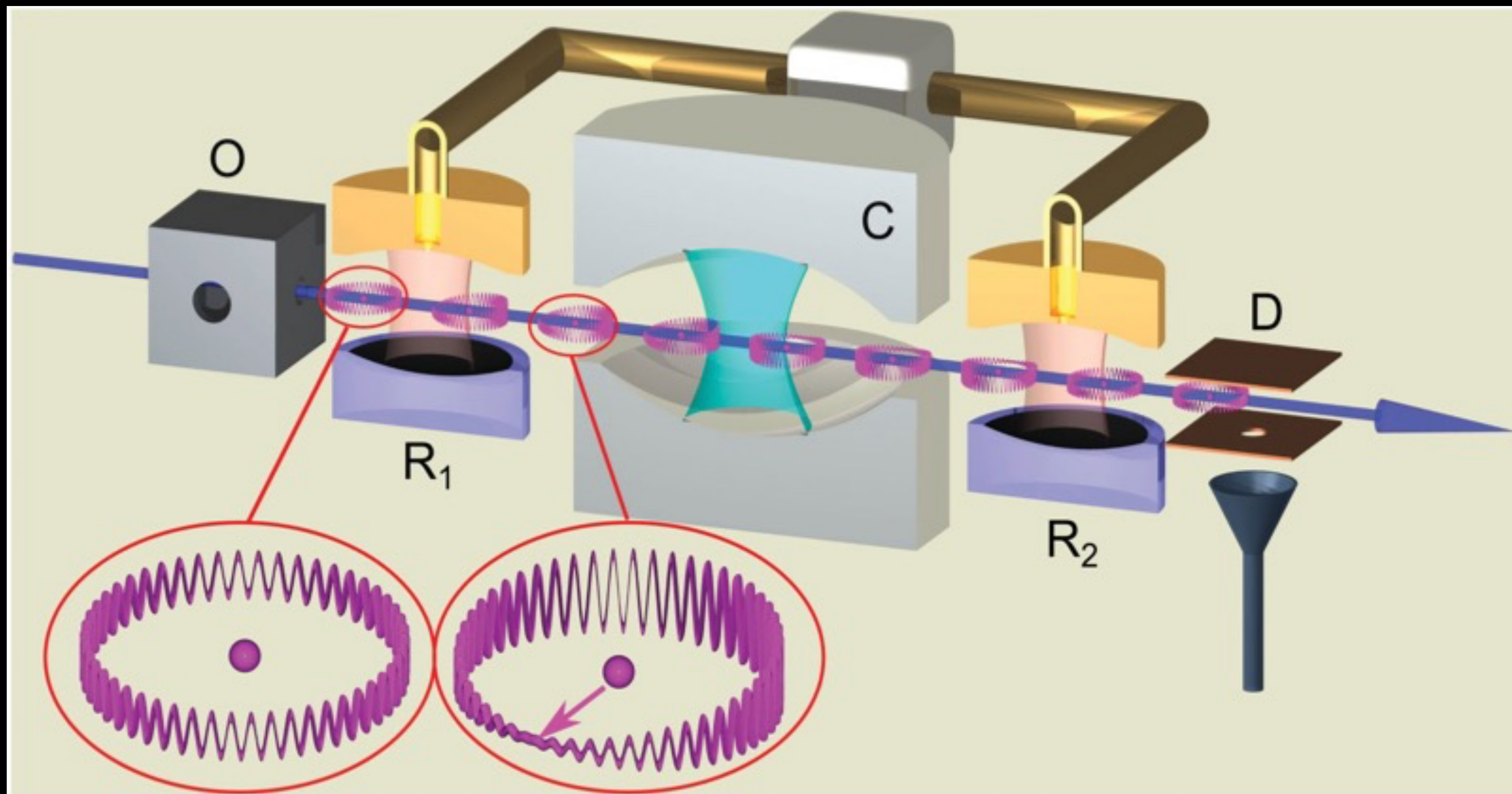
"Explanation" of $P(\underline{u})$ via a finite memory system as hidden cause:



Of course, need to specify the nature of the causation, and of the memory...

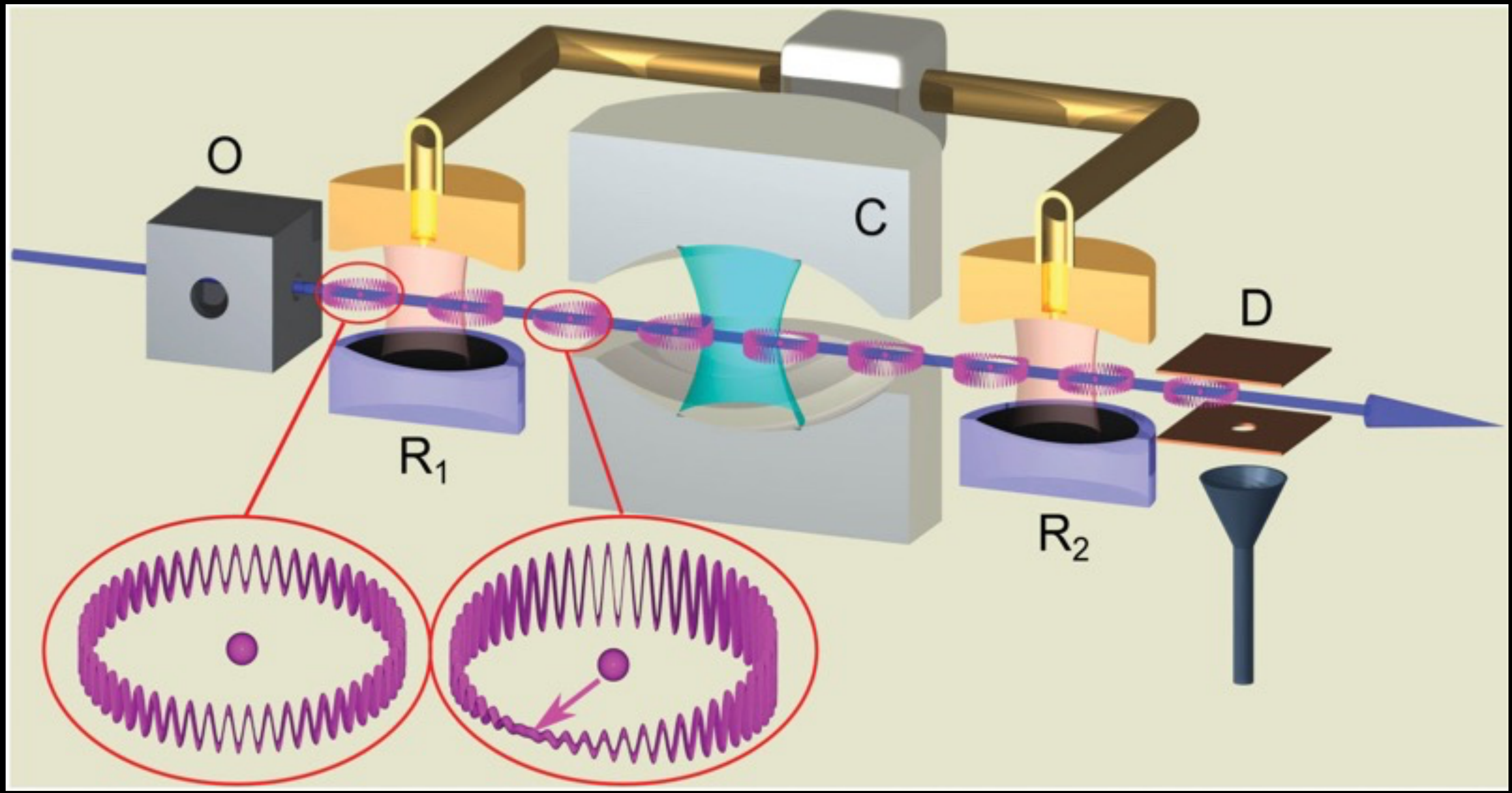
Example: Cavity-atom interaction

[Courtesy of S. Haroche]:



Example: Cavity-atom interaction

[Courtesy of S. Haroche]:



Question: can one infer the quantum nature of the internal mechanism by observing $P(\underline{u})$?

Outline

√. Observations as consequence of a finitary hidden cause (memory)

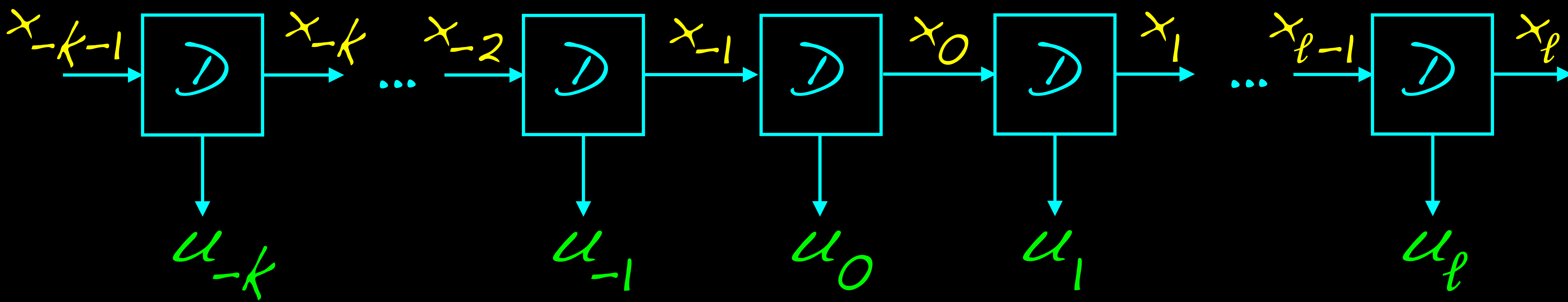
1. Classical, quantum and GPT memory

2. Reconstructing a quasirealisation: low-rank Hankel matrix (completion)

3. Separations: classical $\overset{\checkmark}{\subsetneq}$ quantum $\overset{\checkmark}{\subsetneq}$ GPT

1-a. Classical memory (HMM)

The $x_t \in \mathbb{X}$ are from a finite set of internal states, $\mathcal{D}: \mathbb{X} \to \mathbb{X} \times \mathbb{U}$ is a stochastic map:



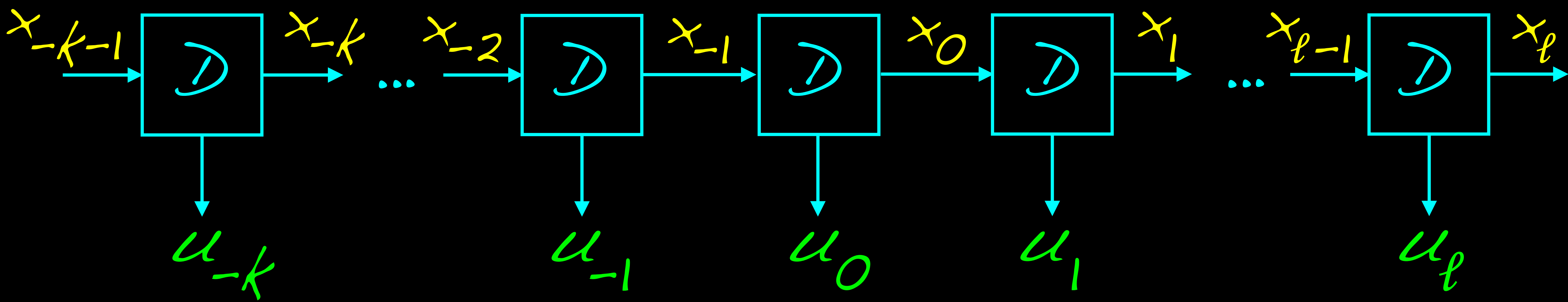
$\mathcal{D}_u : \mathbb{X} \to \mathbb{X}$ are sub-stochastic maps, s.t.

$\bar{\mathcal{D}} = \sum_u \mathcal{D}_u$ is stochastic with stationary distribution π: $\bar{\mathcal{D}}\vec{\mathbf{1}} = \vec{\mathbf{1}}$, $\pi\bar{\mathcal{D}} = \pi$.

$$P(u_1 u_2 \dots u_\ell) = \pi \mathcal{D}_{u_1} \mathcal{D}_{u_2} \cdots \mathcal{D}_{u_\ell} \vec{\mathbf{1}} \qquad \text{(p.r.)}$$

1-6. Quantum memory ($\mathcal{H}$QMM)

The $x_t \in \mathbb{X} = S(\mathcal{H})$ are quantum states on $\mathcal{H}$, and $\mathcal{D}$ is a completely positive instrument:

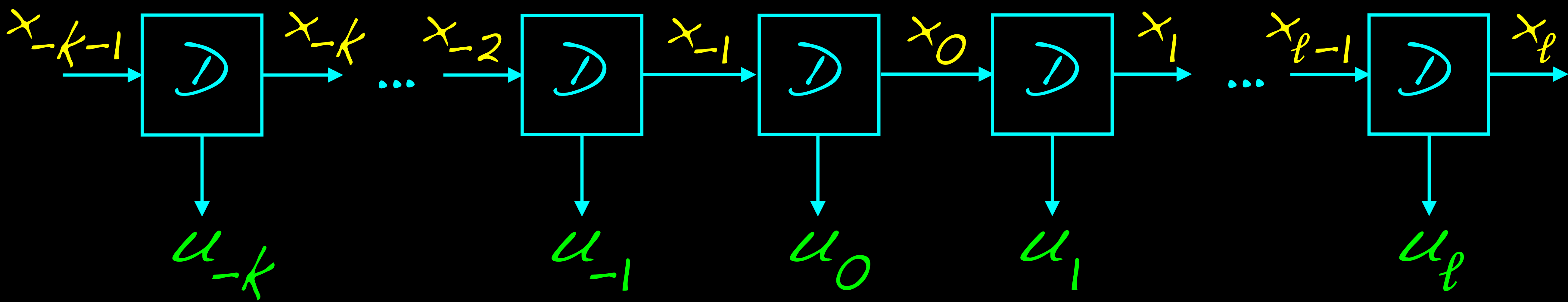


$\mathcal{D}_u : \mathbb{X} \to \mathbb{X}$ are completely positive maps, s.t. $\overline{\mathcal{D}} = \sum_u \mathcal{D}_u$ is unital (cpup) with stationary state ω: $\overline{\mathcal{D}}\mathbb{1} = \mathbb{1}$, $\omega \circ \overline{\mathcal{D}} = \omega$.

$$P(u_1 u_2 \ldots u_\ell) = \omega \circ \mathcal{D}_{u_1} \circ \mathcal{D}_{u_2} \cdots \circ \mathcal{D}_{u_\ell} \mathbb{1} \qquad \text{(c.p.r.)}$$

1-6. Quantum memory ($\mathcal{H}$QMM)

The $x_t \in \mathbb{X} = S(\mathcal{H})$ are quantum states on $\mathcal{H}$, and $\mathcal{D}$ is a completely positive instrument:



$\mathcal{D}_u : \mathbb{X} \to \mathbb{X}$ are completely positive maps, s.t.

$\bar{\mathcal{D}} = \sum_u \mathcal{D}_u$ is unital (cpup) ω

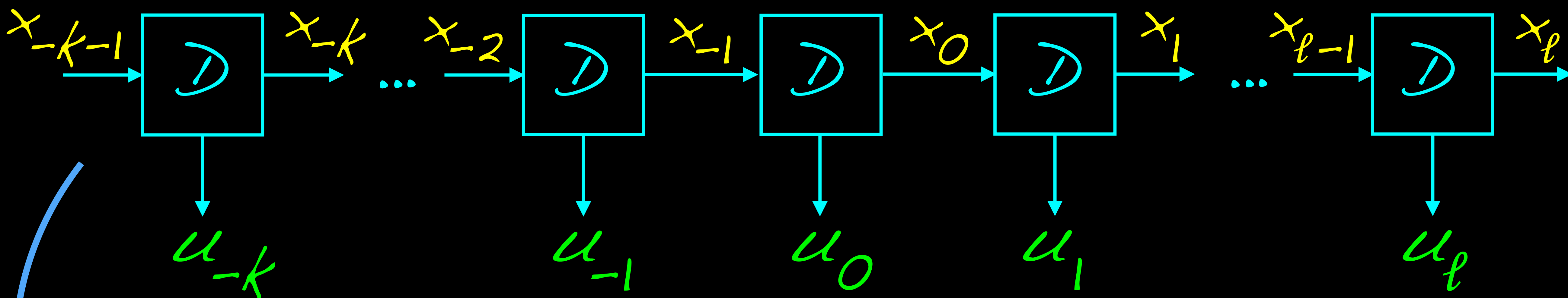
state ω: $\bar{\mathcal{D}}\mathbb{1} = \mathbb{1}$, $\omega \circ \bar{\mathcal{D}} = \omega$.

C*-finitely correlated state

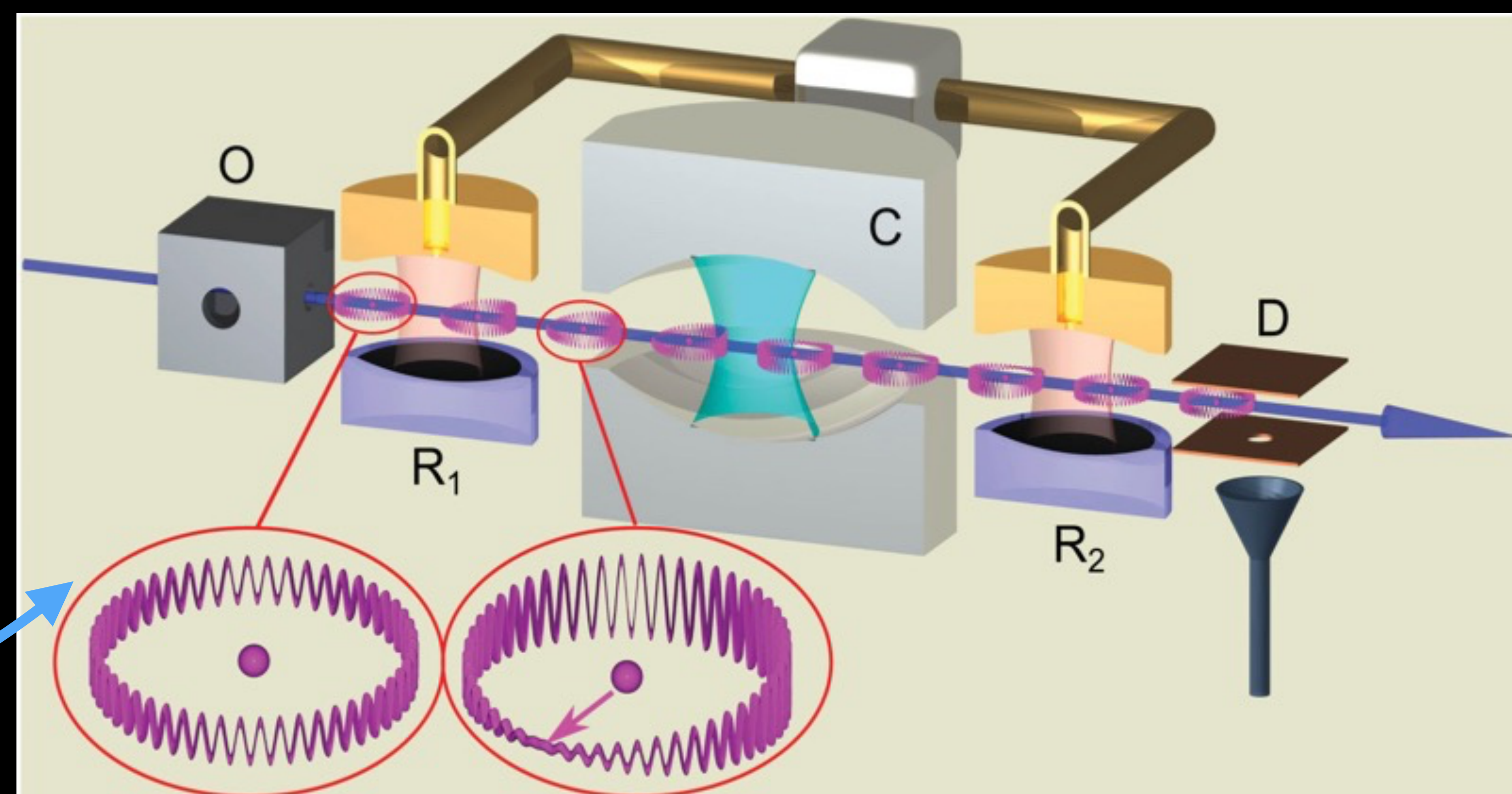
$$P(u_1 u_2 \dots u_\ell) = \omega \circ \mathcal{D}_{u_1} \circ \mathcal{D}_{u_2} \cdots \circ \mathcal{D}_{u_\ell} \mathbb{1}$$ (c.p.r.)

1-6. Quantum memory ($\mathcal{H}$QMM)

The $x_t \in \mathbb{X} = S(\mathcal{H})$ are quantum states on $\mathcal{H}$, and $\mathcal{D}$ is a completely positive instrument:

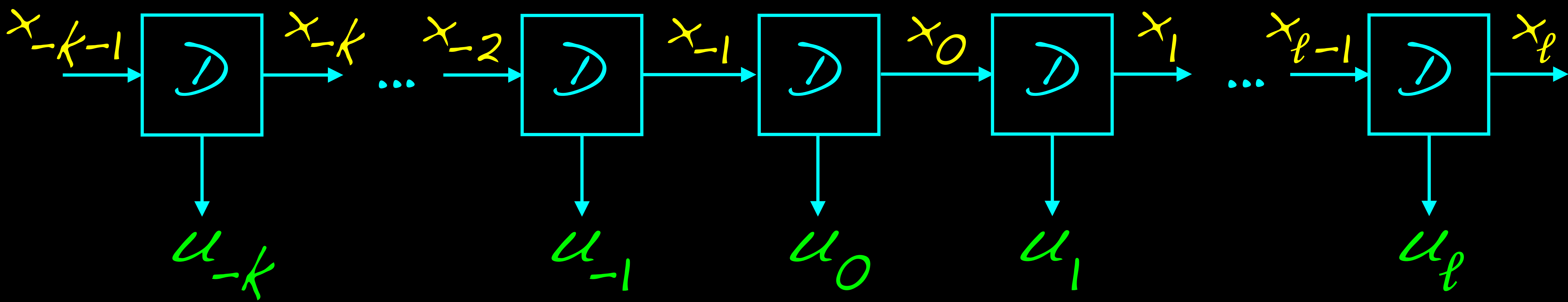


In real life (=in the laboratory):



1-c. General linear structure

The $x_t \in V$ are elements of a (real) vector space, and $\mathcal{D}$ is a collection of linear maps:



$\mathcal{D}_u : V \to V$ are linear maps, $\tau \in V$, $\pi \in V^*$, s.t.

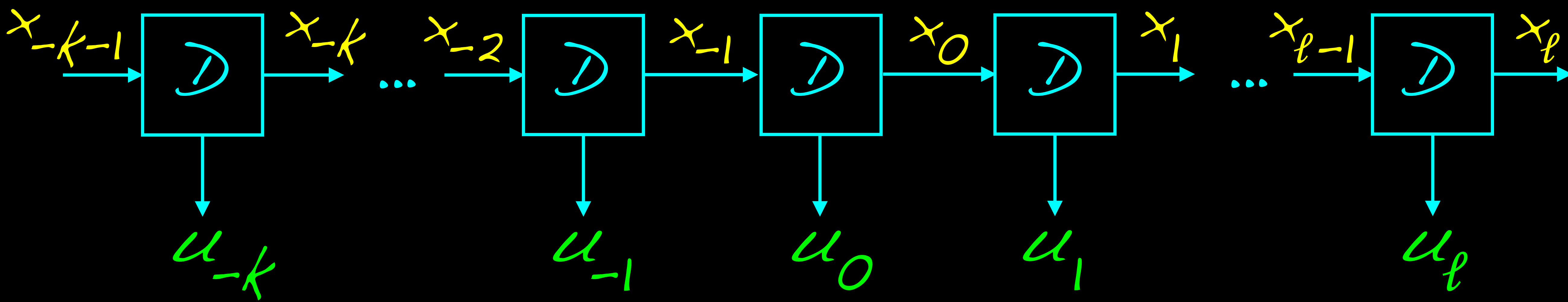
$\bar{\mathcal{D}} = \sum_u \mathcal{D}_u$ preserves both τ and ω:

$\bar{\mathcal{D}}\tau = \tau$, $\pi \circ \bar{\mathcal{D}} = \pi$, as well as $\pi(\tau) = 1$.

$$P(u_1 u_2 \dots u_\ell) = \pi \circ \mathcal{D}_{u_1} \circ \mathcal{D}_{u_2} \cdots \circ \mathcal{D}_{u_\ell} \tau \qquad \text{(gu.r.)}$$

1-c. General linear structure

The $x_t \in V$ are elements of a (real) vector space, and $\mathcal{D}$ is a collection of linear maps:



$\mathcal{D}_u : V \to V$ are linear maps, $\tau \in V$, $\pi \in V^*$, s.t.

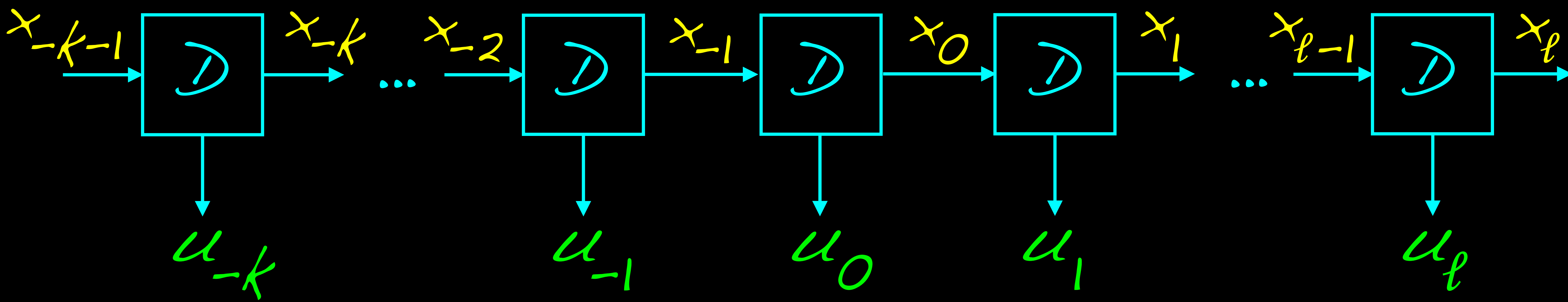
$\bar{\mathcal{D}} = \sum_u \mathcal{D}_u$ preserves both τ a…

$\bar{\mathcal{D}}\tau = \tau$, $\pi \circ \bar{\mathcal{D}} = \pi$, as well as $\pi(\tau$…

$$P(u_1 u_2 \dots u_\ell) = \pi \circ \mathcal{D}_{u_1} \circ \mathcal{D}_{u_2} \cdots \mathcal{D}_{u_\ell} \tau$$ (q.u.r.)

Quasirealisation: general finitely correlated state

1-c. General linear structure



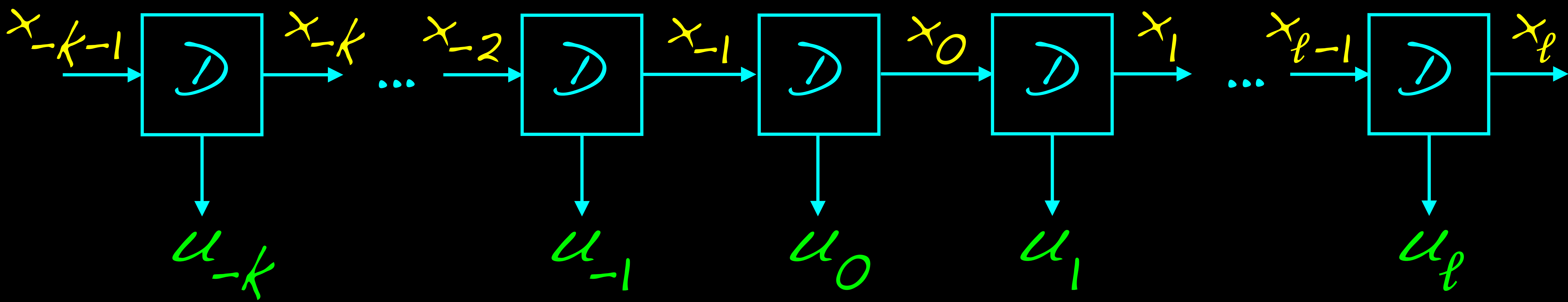
$D_u : V \to V$ are linear maps, $\tau \in V$, $\omega \in V^*$, s.t.

$\bar{D} = \sum_u D_u$ preserves both τ and π, $\pi(\tau)=1$.

$$P(u_1 u_2 \ldots u_\ell) = \pi \circ D_{u_1} \circ D_{u_2} \cdots \circ D_{u_\ell} \tau \qquad \text{(qu.r.)}$$

Unlike classical and quantum case, no a priori guarantee that $P(\underline{u}) \geq 0$.

I-c. General linear structure



$D_u : V \to V$ are linear maps, $\tau \in V$, $\omega \in V^*$, s.t.

$\bar{D} = \sum_u D_u$ preserves both τ and π, $\pi(\tau) = 1$.

$$P(u_1 u_2 \dots u_\ell) = \pi \circ D_{u_1} \circ D_{u_2} \cdots \circ D_{u_\ell} \tau$$ (gu.r.)

Unlike classical and quantum case, no a priori guarantee that $P(\underline{u}) \geq 0$. In fact, checking positivity is undecidable ⚡

[Sontag, J. Comp. Syst. Sci. 11(3):375-381, 1975]

Simple examples: - I.i.d. distributed $u_t \in \mathbb{U}$, i.e. $P = P_1^{\otimes \mathbb{Z}}$ infinite product of single-letter distributions P_1. Requires no, or rather only trivial, memory: $\dim V = 1$.

Simple examples: -I.i.d. distributed $u_t \in \mathbb{U}$, i.e. $P = P_1^{\otimes \mathbb{Z}}$ infinite product of single-letter distributions P_1. Requires no, or rather only trivial, memory: $\dim V = 1$.

-De Finetti distribution $P = \sum_{x \in \mathbb{X}} \pi_x P_x^{\otimes \mathbb{Z}}$ with distinct p.d.'s P_x and $\pi_x > 0$.

Realised as HMM by memorising $x \in \mathbb{X}$ forever, and indeed $\dim V \geq |\mathbb{X}|$ is both sufficient and necessary for a quasirealisation.

Simple examples: -I.i.d. distributed $u_t \in \mathbb{U}$, i.e. $P = P_1^{\otimes \mathbb{Z}}$ infinite product of single-letter distributions P_1. Requires no, or rather only trivial, memory: $\dim V = 1$.

-De Finetti distribution $P = \sum_{x \in \mathbb{X}} \pi_x P_x^{\otimes \mathbb{Z}}$ with distinct p.d.'s P_x and $\pi_x > 0$.

Realised as HMM by memorising $x \in \mathbb{X}$ forever, and indeed $\dim V \geq |\mathbb{X}|$ is both sufficient and necessary for a quasirealisation.

For $|\mathbb{X}| = \infty$: stationary process not realised as HMM, HQMM, or even quasirealisation.

Example. $V = B(\mathbb{C}^2)_{sa} = \mathrm{span}\{\mathbb{1}, X, Y, Z\}$ qubit with $\tau = \mathbb{1}$, $\pi = \frac{1}{2}\mathrm{Tr}$, and the following maps:

$$D_0(A) = \frac{1}{4}\, |0\rangle\langle 0|\, A\, |0\rangle\langle 0|,$$
$$D_1(A) = \frac{1}{4}\, |1\rangle\langle 1|\, A\, |1\rangle\langle 1|,$$
$$D_X(A) = \frac{1}{4}\, \exp(i\alpha X)\, A\, \exp(-i\alpha X),$$
$$D_Z(A) = \frac{1}{4}\, \exp(i\beta Z)\, A\, \exp(-i\beta Z),$$
$$D_T(A) = \frac{1}{4}\, A^T.$$

Example. $V = B(\mathbb{C}^2)_{sa} = \mathrm{span}\{\mathbb{1}, X, Y, Z\}$ qubit with $\tau = \mathbb{1}$, $\pi = \frac{1}{2}\mathrm{Tr}$, and the following maps:

$$D_0(A) = \frac{1}{4} |0\rangle\langle 0| A |0\rangle\langle 0|,$$
$$D_1(A) = \frac{1}{4} |1\rangle\langle 1| A |1\rangle\langle 1|,$$
$$D_X(A) = \frac{1}{4} \exp(i\alpha X) A \exp(-i\alpha X),$$
$$D_Z(A) = \frac{1}{4} \exp(i\beta Z) A \exp(-i\beta Z),$$
$$D_T(A) = \frac{1}{4} A^T.$$

When α/π and β/π are irrational, dynamics explores whole Bloch sphere densely. Four-dim. qu.r., but requires 2 qubits for c.p.r.!

Example. $V = B(\mathbb{C}^2)_{sa} = \mathrm{span}\{\mathbb{1}, X, Y, Z\}$ qubit with $\tau = \mathbb{1}$, $\pi = \frac{1}{2}\mathrm{Tr}$, and the following maps:

$$D_0(A) = \frac{1}{4}\, |0\rangle\langle 0|\, A\, |0\rangle\langle 0|,$$

$$D_1(A) = \frac{1}{4}\, |1\rangle\langle 1|\, A\, |1\rangle\langle 1|,$$

$$D_x(A) = \frac{1}{4}\, \exp(i\alpha X)\, A\, \exp(-i\alpha X),$$

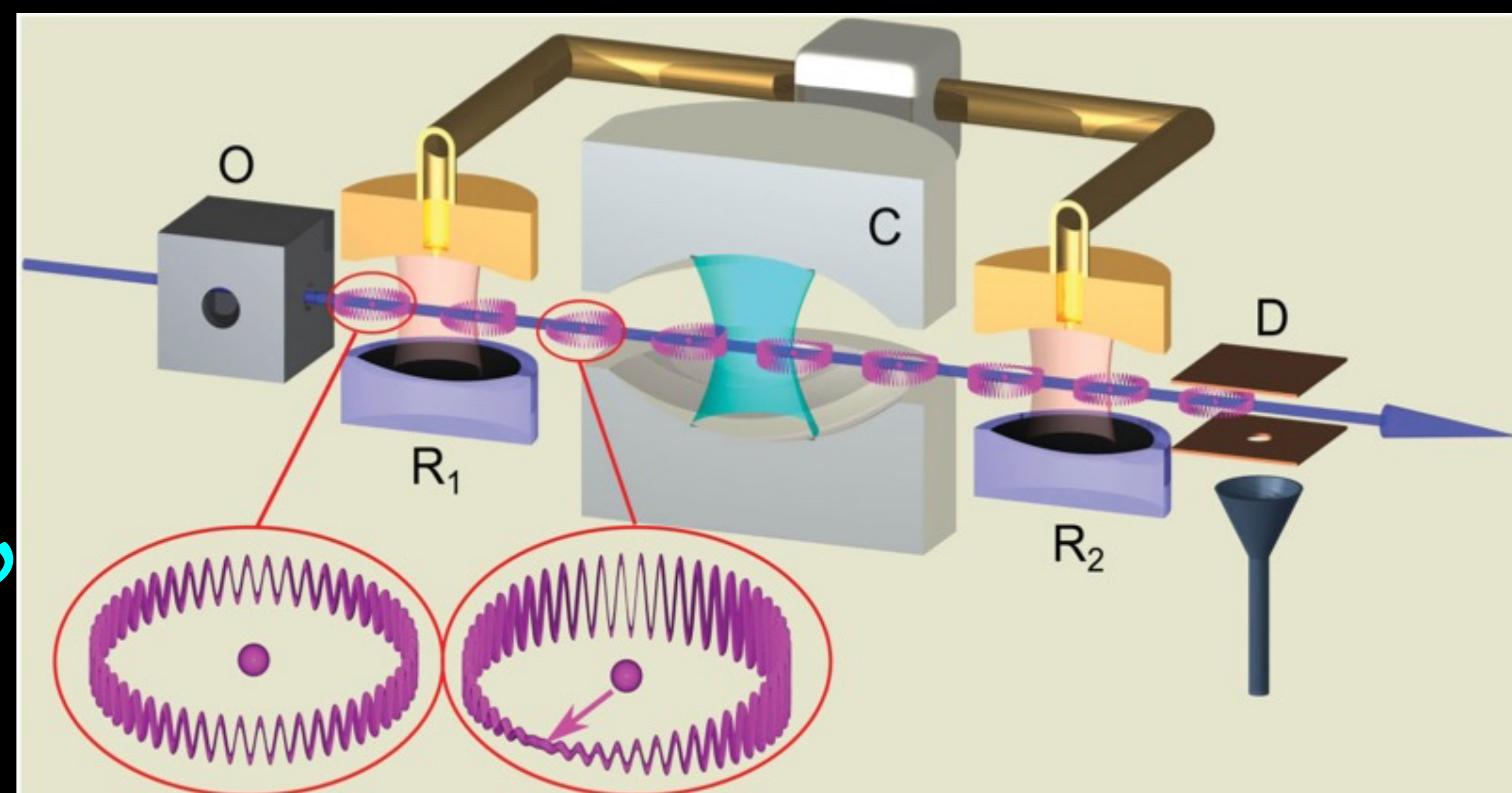
$$D_z(A) = \frac{1}{4}\, \exp(i\beta Z)\, A\, \exp(-i\beta Z),$$

~~$$D_T(A) = \frac{1}{4}\, A^T.$$~~

When α/π and β/π are irrational, dynamics explores whole Bloch sphere densely. Four-dim. qu.r.: HQMM with qubit memory.

Recover the internal mechanism from $P(\underline{u})$?

Quantum application: characterisation of quantum devices - state preparation, gates and measurements - from first principles.
[R. Blume-Kohout et al., 1310.4492] treat system as a black box whose reaction to different interventions we can observe…



Evidently possible only up to linear equivalence, e.g. isometries.

What guarantees positivity of probability?

* $\underline{u} = u_1 u_2 \ldots u_\ell \mapsto \mathcal{D}_{\underline{u}} = \mathcal{D}_{u_1} \circ \mathcal{D}_{u_2} \cdots \circ \mathcal{D}_{u_\ell}$ is semigroup representation.

What guarantees positivity of probability?

* Classical & quantum case: positivity $P(\underline{u}) \geq 0$ enforced by the vector space order.
Generally: Assume we have convex cones $C \subset V$ and $\tilde{C} \subset C' \subset V^*$, s.t. $\tau \in C$, $\pi \in \tilde{C}$, and the cones are preserved by the transformations, i.e. $D_u C \subset C$, $\tilde{C} D_u \subset \tilde{C}$ $\forall u$. Then $P(\underline{u}) \geq 0$.

What guarantees positivity of probability?

* Classical & quantum case: positivity $P(\underline{u}) \geq 0$ enforced by the vector space order.
  Generally: Assume we have convex cones $C \subset V$ and $\tilde{C} \subset C' \subset V^*$, s.t. $\tau \in C$, $\pi \in \tilde{C}$, and the cones are preserved by the transformations, i.e. $D_u C \subset C$, $\tilde{C} D_u \subset \tilde{C}$ $\forall u$. Then $P(\underline{u}) \geq 0$.

Dual cone $C' = \{ f \in V^* : f(x) \geq 0 \;\; \forall x \in C \}$

* Classical & quantum case: positivity $P(\underline{u})\geq 0$ enforced by the vector space order.

Generally: Assume we have convex cones $C \subset V$ and $\tilde{C} \subset C' \subset V^*$, s.t. $\tau \in C$, $\pi \in \tilde{C}$, and the cones are preserved by the transformations, i.e. $D_u C \subset C$, $\tilde{C} D_u \subset \tilde{C}$ $\forall u$. Then $P(\underline{u})\geq 0$.

Conversely: If $P\geq 0$, then such cones exist, e.g. $C = C_{min} = \overline{\text{cone}\{D_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\}}$,

$\tilde{C} = C'_{max} = \overline{\text{cone}\{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}}$.

* Classical & quantum case: positivity $P(\underline{u})\geq 0$ enforced by the vector space order.

Generally: Assume we have convex cones $C \subset V$ and $\tilde{C} \subset C' \subset V^*$, s.t. $\tau\in C$, $\pi\in\tilde{C}$, and the cones are preserved by the transformations, i.e. $D_u C \subset C$, $\tilde{C} D_u \subset \tilde{C}$ $\forall u$. Then $P(\underline{u})\geq 0$.

Conversely: If $P\geq 0$, then such cones exist, e.g. $C = C_{min} = \overline{\text{cone}\{D_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\}}$,

$\tilde{C} = C'_{max} = \overline{\text{cone}\{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}}$.

...not unique, could for instance also take dual cone $\tilde{C} = C'$; call any such C "suitable".

Interpretation: finite-dimensional quasirealisation "explains" time series P by the hidden mechanism of a general probabilistic theory (GPT):

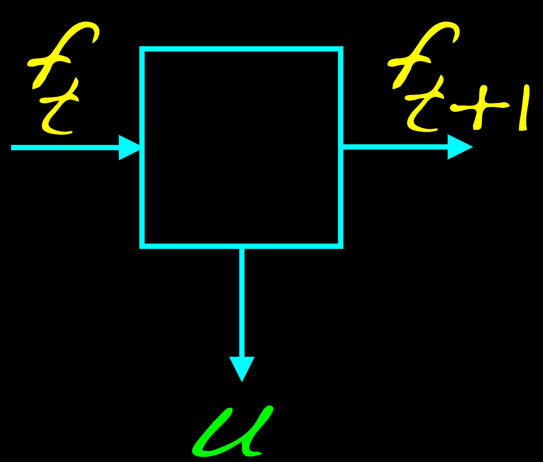
- C and C' are pointed and generating cones;
- $\tau \in int(C)$ and $S := \{f \in C' : f(\tau) = 1\}$ state space;
- $\mathfrak{E} := C \cap (\tau - C)$ "effects" for measurements.

[G. Ludwig & school, 1960s-70s, ...]

Interpretation: finite-dimensional quasirealisation "explains" time series P by the hidden mechanism of a **general probabilistic theory (GPT)**:

- C and C' are pointed and generating cones;
- $\tau \in \mathrm{int}(C)$ and $S := \{f \in C' : f(\tau) = 1\}$ state space;
- $\mathfrak{E} := C \cap (\tau - C)$ "effects" for measurements.

[G. Ludwig & school, 1960s-70s, ...]



$\equiv \Big\{$ $f_t \circ D_u = \Pr\{u|f_t\} f_{t+1}$, relates current & future states, and the output u.

2. Reconstruction of V

* Consider the Hankel-type matrix $\mathcal{H} = (\mathcal{H}_{\underline{u},\underline{v}})$,

with $\mathcal{H}_{\underline{u},\underline{v}} = P(\underline{uv}) = P(u_1 u_2 \ldots u_\ell v_1 v_2 \ldots v_k)$

$= \mathcal{H}_{\varepsilon,\underline{uv}} = \mathcal{H}_{\underline{uv},\varepsilon}$.

2. Reconstruction of V

* Consider the Hankel-type matrix $\mathcal{H} = (\mathcal{H}_{\underline{u},\underline{v}})$, with $\mathcal{H}_{\underline{u},\underline{v}} = P(\underline{uv}) = P(u_1 u_2 \ldots u_\ell v_1 v_2 \ldots v_k)$

$$= \mathcal{H}_{\varepsilon,\underline{uv}} = \mathcal{H}_{\underline{uv},\varepsilon} .$$

* If the process P has a quasirealisation of $\dim V = d$, then

$$\mathcal{H}_{\underline{u},\underline{v}} = (\pi \circ \mathcal{D}_{\underline{u}})(\mathcal{D}_{\underline{v}} \tau),$$

and so $\operatorname{rank} \mathcal{H} \leq d$.

2. Reconstruction of V

* Consider the Hankel-type matrix $\mathcal{H} = (\mathcal{H}_{\underline{u},\underline{v}})$, with $\mathcal{H}_{\underline{u},\underline{v}} = P(\underline{uv}) = P(u_1 u_2 \dots u_\ell v_1 v_2 \dots v_k)$

$$= \mathcal{H}_{\varepsilon,\underline{uv}} = \mathcal{H}_{\underline{uv},\varepsilon} .$$

* If the process P has a quasirealisation of $\dim V = d$, then

$$\mathcal{H}_{\underline{u},\underline{v}} = (\pi \circ \mathcal{D}_{\underline{u}})(\mathcal{D}_{\underline{v}} \tau),$$

and so $\operatorname{rank} \mathcal{H} \leq d$.

* If P has p.r. w/ s states, then $d=s$; if it has c.p.r. w/ Hilbert space dimension t, then $d=t^2$.

* If P has p.r. w/ s states, then $d=s$; if it has c.p.r. w/ Hilbert space dimension t, then $d=t^2$.

Thus: finite rank of $\mathcal{H}$ necessary requirement for the existence of a quasirealisation, and hence of classical or quantum hidden Markov models.

Is it sufficient?

* Consider the Hankel-type matrix $\mathcal{H} = (\mathcal{H}_{\underline{u},\underline{v}})$,
  with $\mathcal{H}_{\underline{u},\underline{v}} = P(\underline{uv}) = P(u_1 u_2 \dots u_\ell v_1 v_2 \dots v_k)$.
  Necessarily of finite rank.
* Conversely, if rank $\mathcal{H} = r < \infty$: There exists a qu.r. ("regular rep.") with $\dim V = r$, which is the minimum. Any other minimal-dim. qu.r. is <u>similar</u> to the regular one, i.e. linearly equivalent.

* Consider the Hankel-type matrix $\mathcal{H}=(\mathcal{H}_{\underline{u},\underline{v}})$, with $\mathcal{H}_{\underline{u},\underline{v}} = P(\underline{uv}) = P(u_1 u_2 \dots u_\ell v_1 v_2 \dots v_k)$. Necessarily of finite rank.
* Conversely, if rank $\mathcal{H} = r < \infty$: There exists a qu.r. ("regular rep.") with dim $V = r$, which is the minimum. Any other minimal-dim. qu.r. is similar to the regular one. [Construction: V = column space of $\mathcal{H}$, and D_u maps $h_{\underline{v}} = \mathcal{H}_{\cdot,\underline{v}}$ to $h_{u\underline{v}} = \mathcal{H}_{\cdot,u\underline{v}} = \mathcal{H}_{\cdot u,\underline{v}}$ – linear because it selects the rows of $h_{\underline{v}}$ with index ending in u; $\tau = h_\varepsilon$, $\pi=(1,0,0,\dots)$. Check that it works...]

Fine, so assume finite rank r of $H=(H_{\underline{u},\underline{v}})$,
i.e. an r-dimensional quasirealisation exists.

Is the process then generated by a finite memory HMM (classical p.r.)?

Fine, so assume finite rank r of $\mathcal{H}=(\mathcal{H}_{\underline{u},\underline{v}})$, i.e. an r-dimensional quasirealisation exists.

Is the process then generated by a finite memory HMM (classical p.r.)?

NO! [Fox/Rubin (1968) and Dharmadhikari/Nadkarni (1970)] provided first examples of processes with finite Hankel rank (actually $r=3$) but requiring infinite classical memory. In fact they're defined as HMM w/ infinite memory. Exploits spectral information from Perron-Frobenius theory.

Alright: assume finite rank r of $\mathcal{H}=(\mathcal{H}_{\underline{u},\underline{v}})$, i.e. an r-dimensional quasirealisation exists.

Is the process then generated by a finite memory HQMM (quantum c.p.r.)?

(Asked by Fannes/Nachtergaele/Werner [CMP 144:443-490 (1992)] for general finitely correlated states.)

Alright: assume finite rank r of $\mathcal{H}=(\mathcal{H}_{\underline{u},\underline{v}})$, i.e. an r-dimensional quasirealisation exists.

Is the process then generated by a finite memory HQMM (quantum c.p.r.)?

(Asked by Fannes/Nachtergaele/Werner [CMP 144:443-490 (1992)] for general finitely correlated states.)

Find: Fox/Rubin's and Dharmadhikari/Nadkarni's processes have HQMMs (with qutrits).

[M. Fanizza/J. Lumbreras/AW, arXiv:2209.11225]

3. Classical $\subsetneq$ quantum $\subsetneq$ GPT

Minimal-dimensional quasirealisation of a process is unique, and isomorphic to the regular representation from $\mathcal{H}$, $\dim V = r$.

3. Classical $\subsetneq$ quantum $\subsetneq$ GPT

Minimal-dimensional quasirealisation of a process is unique, and isomorphic to the regular representation from $\mathcal{H}$, $\dim V = r$.

Fact: Given any quasirealisation V, then the regular one is obtained by going to quotient $V_0 =: \mathrm{span}(C_{min}) / \ker(C'_{max})$.

3. Classical $\subsetneq$ quantum $\subsetneq$ GPT

Minimal-dimensional quasirealisation of a process is unique, and isomorphic to the regular representation from $\mathcal{H}$, $\dim V = r$.

Fact: Given any quasirealisation V, then the regular one is obtained by going to quotient $V_0 =: \mathrm{span}(C_{min}) / \ker(C'_{max})$.

For the cone C (classical, quantum or GPT), this means intersecting it with $\mathrm{span}(C_{min})$, and factoring out $\ker(C'_{max})$.

Recall cones:

Given convex cones $C \subset V$ and $\tilde{C} \subset C' \subset V^*$, s.t. $\tau \in C$, $\pi \in C$, and the cones are preserved by the transformations, i.e. $D_u C \subset C$, $\tilde{C} D_u \subset \tilde{C}$ for all u. Then $P(\underline{u}) \geq 0$.

Conversely: If $P \geq 0$, then such cones exist, e.g. $C = C_{min} = \overline{\text{cone}\{D_{\underline{u}} \tau : \underline{u} \in \mathbb{U}^*\}}$,

$\tilde{C} = C'_{max} = \overline{\text{cone}\{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}}$.

But not unique: many cones between C_{min} and C_{max} are suitable: $C_{min} \subset C \subset C_{max}$.

(Also, of course, C has to be stable under the maps D_u!)

A HMM (p.r.) has the cone of non-negative vectors; this gives rise to polyhedral cones C & C' in the regular representation.

A HMM (p.r.) has the cone of non-negative vectors; this gives rise to polyhedral cones C & C' in the regular representation.

A HQMM (c.p.r.) has cone of semidefinite matrices; this gives rise to semidefinite representable (SDR) cones C & C' in the regular representation:

$$C = \{x=(x_1,\dots,x_d) : \exists x_{d+1},\dots x_e \sum_{i=1}^{e} x_i R_i \geq 0\},$$

for certain DxD-matrices R_i.

Example. $V = B(\mathbb{C}^2)_{sa} = \mathrm{span}\{\mathbb{1}, X, Y, Z\}$ qubit with $\tau = \mathbb{1}$, $\pi = \frac{1}{2}Tr$, and the following maps:

$$D_0(A) = \frac{1}{4}\, |0\rangle\langle 0|\, A\, |0\rangle\langle 0|,$$
$$D_1(A) = \frac{1}{4}\, |1\rangle\langle 1|\, A\, |1\rangle\langle 1|,$$
$$D_x(A) = \frac{1}{4}\, \exp(i\alpha X)\, A\, \exp(-i\alpha X),$$
$$D_z(A) = \frac{1}{4}\, \exp(i\beta Z)\, A\, \exp(-i\beta Z),$$
$$D_T(A) = \frac{1}{4}\, A^T.$$

When α/π and β/π are irrational, dynamics explores whole Bloch sphere densely. Four-dim. qu.r., but requires 2 qubits for c.p.r.!

In the previous example, $C_{min} = C_{max}$, hence C is unique, and it's not polyhedral: cone over Bloch sphere. Thus, this process has no (finite) classical realisation.

Polyhedral cone between C_{min} and C_{max} necessary for cl. realisation. Sufficient?

[Cf. however Dharmadhikari/Nadkarni]

SDR cone between C_{min} and C_{max} necessary for existence of a quantum realisation.

Sufficient? (...)

SDR cone between C_{min} and C_{max} necessary for existence of a quantum realisation. Sufficient? (...)

Thm. [M. Fanizza/J. Lumbreras/AW, 2209.11225]: ∃ process P with Hankel rank $\mathcal{H} = 3$ and $C_{min} = C_{max}$ transcendental, whereas SDR cones are semi-algebraic. Thus, P has no $\mathcal{H}$QMM.

SDR cone between C_{min} and C_{max} necessary for existence of a quantum realisation. Sufficient? (...)

Thm. [M. Fanizza/J. Lumbreras/AW, 2209.11225]: ∃ process P with Hankel rank $\mathcal{H} = 3$ and $C_{min} = C_{max}$ transcendental, whereas SDR cones are semi-algebraic. Thus, P has no $\mathcal{H}$QMM.

Answers open question of Fannes/Nachtergaele/Werner [CMP 144:443-490 (1992)] :-)

Example: P has three symbols, 0, 1, 2.

We give directly its quasirealisation:

$V=\mathbb{R}^3$; let $a>1>b>0$ such that ln a and ln b are linearly independent over the rationals.

$$D_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln a & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln b & 1 \end{bmatrix},$$

$$D_0 = m_0 \mu_0^T, \text{ with } m_0 = \begin{bmatrix} m_{01} \\ m_{02} \\ m_{03} \end{bmatrix},$$

$$\mu_0^T = [\mu_{01} \; \mu_{02} \; \mu_{03}]$$

Example: P has three symbols, 0, 1, 2.

We give directly its quasirealisation:

$V=\mathbb{R}^3$; let $a>1>b>0$ such that $\ln a$ and $\ln b$ are linearly independent over the rationals.

$$D_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln a & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln b & 1 \end{bmatrix},$$

$$D_1^s D_2^t = \begin{bmatrix} a^s b^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s(\ln a)+t(\ln b) & 1 \end{bmatrix} \text{ for } s, t \in \mathbb{N}$$

Example: P has three symbols, 0, 1, 2.
We give directly its quasirealisation:
$V=\mathbb{R}^3$; let $a>1>b>0$ such that $\ln a$ and $\ln b$ are linearly independent over the rationals.

$$D_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln a & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln b & 1 \end{bmatrix},$$

$$D_1^s D_2^t = \begin{bmatrix} a^s b^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s(\ln a)+t(\ln b) & 1 \end{bmatrix} \text{ for } s, t \in \mathbb{N}$$

Example: P has three symbols, 0, 1, 2.
We give directly its quasirealisation:
$V=\mathbb{R}^3$; let $a>1>b>0$ such that $\ln a$ and $\ln b$ are linearly independent over the rationals.

$$D_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln a & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \ln b & 1 \end{bmatrix},$$

$$D_1^s D_2^t = \begin{bmatrix} e^x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix}, \text{ with } x \in \mathbb{R} \text{ dense!}$$

Example (cont'd): D_0 is a "reset" operation (making P a "birth process"), so can write

$$C'_{max} = \text{cone}\{[\mu_{01}e^x \;\; \mu_{02}+\mu_{03}x \;\; \mu_{03}] : x\in\mathbb{R}\}$$

$$C_{min} = \text{cone}\{[m_{01}e^x \;\; m_{03} \;\; m_{02}+m_{03}x]^T : x\in\mathbb{R}\}$$

Fact: C_{max} is of the same form as C_{min}, only with different parameters. One can choose D_0 such that $C_{min}=C_{max}=C$.

Example (cont'd): D_0 is a "reset" operation (making P a "birth process"), so can write

$$C'_{max} = \text{cone}\{[\mu_{01}e^{x}\ \ \mu_{02}+\mu_{03}x\ \ \mu_{03}] : x \in \mathbb{R}\}$$

$$C_{min} = \text{cone}\{[m_{01}e^{x}\ \ m_{03}\ \ m_{02}+m_{03}x]^{T} : x \in \mathbb{R}\}$$

Fact: C_{max} is of the same form as C_{min}, only with different parameters. One can choose D_0 such that $C_{min}=C_{max}=C$.

In that case, a suitable positive linear combination of D_0, D_1, D_2 has right fixed point τ in int(C), and left fixed point π in int(C'). This is the sought-after qu.r. (...)

Example gives rise to the exponential cone

$$K_{exp} = \{(x,y,z) : x/y \geq e^{z/y},\ x,y,z \geq 0\},$$

and it works for us because that is a transcendental shape.

Example gives rise to the exponential cone

$$K_{exp} = \{(x,y,z) : x/y \geq e^{z/y},\ x,y,z \geq 0\},$$

and it works for us because that is a transcendental shape.

More examples from power cone $(0<t<1)$

$$K_t = \{(x,y,z) : x^t y^{1-t} \geq |z|,\ x,y \geq 0,\ z \in \mathbb{R}\},$$

which is transcendental iff t is irrational.

As before we can design a reset map and two invertible maps, which latter act densely transitive on the cone's extremal rays. And we can engineer $C_{min} = C_{max}$, too.

4. Further thoughts

- Exhibited a process (FCS) without a quantum realisation (i.e. it is not C^*-FCS). However, that is an asymptotic statement, every finite block of the chain is c.p. representable. To approximate P on n sites to error ε, how large a virtual dimension t do we need? For fixed t and $n \to \infty$, does ε become arbitrarily small, or is it bounded away from 0, or does even converges to 1?

4. Further thoughts

Extend to genuinely quantum case, i.e. a chain of non-commutative spin C^*-algebras:

- Have a generalisation of regular (minimum dim.) representation for finitely corr. states
- Rather than a vector space order on V and positive elements and maps, necessary and sufficient structure is an operator system, i.e. consistent orders on $V \otimes M_n$, and maps are completely positive...

[Fannes/Nachtergaele/Werner, CMP 144:443-490 (1992)]

4. Further thoughts

Finitely correlated state on a chain of non-commutative spin C^*-algebras:

- In fact, the finitely correlated state itself gives us two extreme o.s., where the cones $(V \otimes M_n)_+$ are either all as small or all as large as they can be.
- Exponential and power cones have matrix generalisations; perhaps suitable for new variational classes of FCS? Need cp maps!

[Fanizza et al., work in progress]

4. Further thoughts

Finitely correlated state on a chain of non-commutative spin C^*-algebras:

- Examples of a FCS that are not C^*-FCS are highly mixed (they're p.d.'s). So what about pure ones?
- Note that C^*-FCS always have C^*-FCS purifications. Do our example FCS have FCS purifications?

[Fanizza et al., work in progress]

4. Questions, questions, questions

- Low-rank completion of the Hankel matrix with noisy data? Cf. [Fanizza/Galke/Lumbreras/Rouzé/AW, arXiv:2312.07516]
- How to find a quantum model just from the regular representation (assuming one exists)?
- Can these exponential and power cones be useful? Note that dual cone is of the same kind, so perhaps good for convex optimisation. Interesting class of GPTs?

=Additional material=

Ogni scarrafon' è
bell' a mamma suja

5. Removing redundancy: quotients

If your model is not minimal, still useful, assuming it has a suitable cone $C \subset V$.

Redundancy...

5. Removing redundancy: quotients

If your model is not minimal, still useful, assuming it has a suitable cone $C \subset V$.

Redundancy: $W = \mathrm{span}\{D_{\underline{u}} \tau : \underline{u} \in \mathbb{U}^*\} \subset V$,

Reachable space; might as well go to W, with cone $C \cap W$...

5. Removing redundancy: quotients

If your model is not minimal, still useful, assuming it has a suitable cone $C \subset V$.

Redundancy: $W = \mathrm{span}\{D_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\} \subset V$,

$K = \{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^\perp \subset V$.

Null space; $C \cap K = 0$, so we may factor out K...

Reachable space; might as well go to W, with cone $C \cap W$...

5. Removing redundancy: quotients

If your model is not minimal, still useful, assuming it has a suitable cone $C \subset V$.

Redundancy: $\mathcal{W} = \mathrm{span}\{\mathcal{D}_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\} \subset V$,

$$K = \{\pi \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset V.$$

$V_0 := \mathcal{W}/K$,

$C_0 := (C \cap \mathcal{W})/K = \{w + K : w \in C \cap \mathcal{W}\}$,

$\tau_0 := \tau + K$, $\pi_0 := \pi/K$, $\mathcal{D}^0_{\underline{u}} := \mathcal{D}_{\underline{u}}/K$; well-defined because of $\pi(K) = 0$, $\mathcal{D}_{\underline{u}}\mathcal{W} \subset \mathcal{W}$, $\mathcal{D}_{\underline{u}} K \subset K$.

5. Removing redundancy: quotients

If your model is not minimal, still useful, assuming it has a suitable cone $C \subset V$.

Redundancy: $W = \mathrm{span}\{D_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\} \subset V$,

$K = \{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset V$.

$V_0 := W/K$,

$C_0 := (C \cap W)/K = \{w + K : w \in C \cap W\}$,

$\tau_0 := \tau + K$, $\pi_0 := \pi/K$, $D^0_u := D_u/K$.

Always a minimal-dim. qu.r., hence is isomorphic to regular, and cone C_0 is suitable.

Redundancy: $\mathcal{W} = \operatorname{span}\{D_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\} \subset V$,

$$K = \{\pi D_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset V.$$

$V_0 := \mathcal{W}/K$,

$C_0 := (C \cap \mathcal{W})/K = \{w + K : w \in C \cap \mathcal{W}\}$,

$\tau_0 := \tau + K$, $\pi_0 := \pi/K$, $D^0_{\underline{u}} := D_{\underline{u}}/K$.

Classical model, i.e. $V = \mathbb{R}^d$, $C = \mathbb{R}^d_{\geq 0}$, $\tau = (1, \ldots, 1)^\top$,

π a probability row vector.

Redundancy: $\mathcal{W} = \mathrm{span}\{ \mathcal{D}_{\underline{u}}\tau \ : \ \underline{u} \in \mathbb{U}^* \} \subset V$,

$K = \{ \pi \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^* \}^{\perp} \subset V$.

$V_0 := \mathcal{W}/K$,

$C_0 := (C \cap \mathcal{W})/K = \{ \omega + K \ : \ \omega \in C \cap \mathcal{W} \}$,

$\tau_0 := \tau + K$, $\pi_0 := \pi/K$, $\mathcal{D}^0_u := \mathcal{D}_u/K$.

Classical model, i.e. $V = \mathbb{R}^d$, $C = \mathbb{R}^d_{\geq 0}$, $\tau = (1, \ldots, 1)^\top$,

π a probability row vector.

C_0 is then a polyhedral cone and every such cone arises in the above way (Fourier-Motzkin elemination). Guaranteed:

$d \leq$ #extremal rays of C, sometimes best.

5'. Quotient of a quantum model

Quantum model, i.e. $V = B(\mathcal{H})_{sa}$, $C = B(\mathcal{H})_{\geq 0}$, $\tau = \mathbb{1}$, $\pi = \omega$ quantum state, $\mathcal{D}_u$ are cp maps.

Once constructed $K \cap W \subset W \subset V$: $C \cap W$ is an operator system, $C_0 = (C \cap W)/K$ a quotient operator system; the $\mathcal{D}_u^0$ preserve C, in fact cp maps in the operator system sense.

[Farenick/Paulsen, Math. Scand. 111:210-243, 2012]

5'. Quotient of a quantum model

Quantum model, i.e. $V=B(\mathcal{H})_{sa}$, $C=B(\mathcal{H})_{\geq 0}$, $\tau=\mathbb{1}$, $\pi=\omega$ quantum state, $\mathcal{D}_u$ are cp maps.

Once constructed $K\cap W \subset W \subset V$: $C\cap W$ is an operator system, $C_0 = (C\cap W)/K$ a quotient operator system; the $\mathcal{D}_u^0$ preserve C, in fact cp maps in the operator system sense.

[Farenick/Paulsen, Math. Scand. 111:210-243, 2012]

Membership in the cone is an SDP: semi-definite condition of a finite-size matrix with existential real variables.

SDR operator systems:

$\mathbb{1} \in \mathcal{W} = \mathrm{span}\{\mathcal{D}_{\underline{u}}\mathbb{1} : \underline{u} \in \mathbb{U}^*\} = B(\mathcal{H})_{sa}$,

$K = \{\omega \circ \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset B(\mathcal{H})_{sa}$.

Vector space and positive cone:

$V_0 := \mathcal{W}/K$,

$C_0 := (B(\mathcal{H})_{\geq 0} \cap \mathcal{W})/K = \{w + K : w \in B(\mathcal{H})_{\geq 0} \cap \mathcal{W}\}$.

[Farenick/Paulsen, Math. Scand. 111:210–243, 2012]

SDR operator systems:

$\mathbb{1} \in \mathcal{W} = \mathrm{span}\{\mathcal{D}_{\underline{u}}\mathbb{1} : \underline{u} \in \mathbb{U}^*\} = B(\mathcal{H})_{sa}$,

$K = \{\omega \circ \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset B(\mathcal{H})_{sa}$.

Vector space and positive cone:

$V_0 := \mathcal{W}/K$,

$C_0 := (B(\mathcal{H})_{\geq 0} \cap \mathcal{W})/K = \{\omega + K : \omega \in B(\mathcal{H})_{\geq 0} \cap \mathcal{W}\}$.

Operator system lifts this to $V_0 \otimes B(\mathbb{C}^n)_{sa}$:

$C_n := (B(\mathcal{H} \otimes \mathbb{C}^n)_{\geq 0} \cap \mathcal{W} \otimes B(\mathbb{C}^n)_{sa})/K \otimes \mathbb{1}$

[Farenick/Paulsen, Math. Scand. 111:210–243, 2012]

SDR operator systems:

$\mathbb{1} \in \mathcal{W} = \mathrm{span}\{\mathcal{D}_{\underline{u}}\mathbb{1} : \underline{u} \in \mathbb{U}^*\} = B(\mathcal{H})_{sa}$,

$K = \{\omega \circ \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}^{\perp} \subset B(\mathcal{H})_{sa}$.

Vector space and positive cone:

$V_0 := \mathcal{W}/K$,

$C_0 := (B(\mathcal{H})_{\geq 0} \cap \mathcal{W})/K = \{\omega + K : \omega \in B(\mathcal{H})_{\geq 0} \cap \mathcal{W}\}$.

Operator system lifts this to $V_0 \otimes B(\mathbb{C}^n)_{sa}$:

$C_n := \left(B(\mathcal{H} \otimes \mathbb{C}^n)_{\geq 0} \cap \mathcal{W} \otimes B(\mathbb{C}^n)_{sa}\right)/K \otimes \mathbb{1}$

CP maps: $(\mathcal{D}_u \otimes id) C_n \subset C_n$ for all u and n.

[Farenick/Paulsen, Math. Scand. 111:210–243, 2012]

But the $\mathcal{D}_{\mathcal{U}}^{o}$ remember more than just being cp in the operator system. Indeed,

$$\mathcal{D}_{\mathcal{U}}^{o} \in \mathcal{P} := \{\Lambda/K : \Lambda \text{ cp on } B(\mathcal{H}),$$

$$\Lambda(\mathcal{W}) \subset \mathcal{W},\ \Lambda(K) \subset K\} \subset End(V_0),$$

which is itself an SDR cone.

But the $\mathcal{D}_U^0$ remember more than just being cp in the operator system. Indeed,

$$\mathcal{D}_U^0 \in \mathcal{P} := \{\Lambda/K : \Lambda \text{ cp on } B(\mathcal{H}), \Lambda(W)\subset W, \Lambda(K)\subset K\} \subset End(V_0),$$

which is itself an SDR cone. Maybe you don't find it too pretty...it took us a while, too, to see its beauty :-)

But the $\mathcal{D}_U^0$ remember more than just being cp in the operator system. Indeed,

$$\mathcal{D}_U^0 \in \mathcal{P} := \{\Lambda/K : \Lambda \text{ cp on } B(\mathcal{H}), \Lambda(\mathcal{W})\subset\mathcal{W}, \Lambda(K)\subset K\} \subset End(V_0),$$

which is itself an SDR cone. Maybe you don't find it too pretty...it took us a while, too, to see its beauty :-)

$\mathcal{P}=\mathcal{P}(\mathcal{W},K) \subset CP(V_0)$, and in general the inclusion is strict!

[Equality by Arveson's extension theorem for $K=0$ or $\mathcal{W}=B(\mathcal{H})_{sa}$]

6. Reconstructing the vector order?

Task: Find a suitable cone C for the qu.r. $(V, \tau, \pi, \mathcal{D}_u)$, ideally a "nice" one...

Necessarily, $C_{min} \subset C \subset C_{max}$, with (recall)

$$C_{min} = \overline{\mathrm{cone}\{\mathcal{D}_{\underline{u}}\tau \,:\, \underline{u} \in \mathbb{U}^*\}},$$

$$C_{max} = \mathrm{cone}\{\pi \mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}'.$$

6. Reconstructing the vector order?

Task: Find a suitable cone C for the qu.r. $(V,\tau,\pi,\mathcal{D}_u)$, ideally a "nice" one...

Necessarily, $C_{min} \subset C \subset C_{max}$, with (recall)

$$C_{min} = \overline{\mathrm{cone}\{\mathcal{D}_{\underline{u}}\tau : \underline{u} \in \mathbb{U}^*\}},$$

$$C_{max} = \mathrm{cone}\{\pi\mathcal{D}_{\underline{u}} : \underline{u} \in \mathbb{U}^*\}'.$$

Can we choose C polyhedral or SDR?

Difficulty of course that C has to be preserved by the $\mathcal{D}_u$; note that C_{min} & C_{max} satisfy this automatically.

6. Reconstructing the vector order?

Instructive special case: $C = C_{min} = C_{max}$, ruling out a classical model if that is not a polyhedral cone. [Cf. example, where this happens with C=cone over a Bloch sphere.]

6. Reconstructing the vector order?

Instructive special case: $C = C_{min} = C_{max}$, ruling out a classical model if that is not a polyhedral cone. [Cf. example, where this happens with C=cone over a Bloch sphere. And the other example, where C is unique and not SDR, in fact transcendental; provides a process generated by a finite GPT, but w/o quantum realisation.]