Conditional entropy and information of quantum processes

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Motivation

- Long quest to understand quantum processes.
- Information theory is powerful tool in analyzing potential of quantum systems.
- Could possibly unravel novel information processing features of quantum processes.

Queries

- How novel would the conditional entropy of quantum processes be when compared with the conditional entropy of states?
- Can the conditional entropy of bipartite quantum processes be strictly lower than the least possible values of entropy of quantum processes?
- If yes, what would that signature property for quantum processes be?

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Quantum states and processes

Quantum states

State of a quantum system contains the description of the system. It is given by a density operator (positive semidefinite operator with unit trace) defined on the Hilbert space associated with the system.

Quantum channels

The physical transformation of quantum states are given by a quantum channel. It is completely positive, trace-preserving map. For any quantum channel $\mathcal{N}_{A\to B}$, there exists a unitary operator $U_{AE'\to BE}$ such that

$$\mathcal{N}_{A\to B}(\cdot) = \mathrm{tr}_E[U_{AE'\to BE}(\cdot \otimes |0\rangle\langle 0|_{E'})(U_{AE'\to BE})^\dagger].$$

The physical transformation of a quantum channel is given by quantum superchannel, e.g., $\Theta(\mathcal{N}_{A\to B}) = \mathcal{Q}_{RB\to D} \circ \mathcal{N}_{A\to B} \circ \mathcal{P}_{C\to RA} = \mathcal{N}'_{C\to D}$.

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Sneak peek 1

	Entropy	Conditional entropy
random variables	$0 \leq H(X) \leq \log \lvert X \rvert$	$0 \leq H(X \vert Y) \leq \log \lvert X \rvert$
quantum states	$0 \leq S(A)_\rho \leq \log \lvert A \rvert$	$-\log \lvert A \rvert \leq S(A \vert B)_\rho \leq \log \lvert A \rvert$
quantum channels	$-\log \lvert A \rvert \leq S[A]_\mathcal{N} \leq \log \lvert A \rvert$ [GW21]	$-\log \lvert A' \rvert \lvert B' \rvert \lvert B \rvert \leq S[A \vert B]_\mathcal{N} \leq \log \lvert A \rvert$

Sneak peek 2

Conditional entropy: quantum states Vs quantum channels

$S(A)_\rho$ achieves minimal value 0 iff the state ρ_A is pure [vN32].

$S[A]_\mathcal{N}$ achieves minimal value $-\log\min\{|A'|,|A|\}$ iff the channel $\mathcal{N}_{A'\to A}$ is isometric [GW21].

$S(A|B)_\rho < 0 \implies$ state ρ_{AB} is entangled [CA97].

$S[A|B]_\mathcal{N} < -\log|A| \implies$ channel $\mathcal{N}_{A'B'\to AB}$ is signaling from $A' \to B$.

$S(A|B) < 0$ for all pure entangled states ρ_{AB} [CA97, HOW05].

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$S(A|BC)_\rho \leq S(A|B)_\rho$ for states ρ_{AB} [LR73].

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Conditional entropy of a channel: Axioms

Axioms

For any conditional entropy function $f[A|B]_{\mathcal{N}}$ of a bipartite channel $\mathcal{N}_{A'B'\to AB}$, desired properties are

- Monotonicity: Nondecreasing under the action of local pre-processing and post-processing with random unitary channels on A', A systems and any pre-processing and post-processing on B', B systems.
- Conditioning shouldn't increase entropy, $f[A|B]_{\mathcal{N}} \leq f[A]_{\mathcal{N}}$.
- Normalization: For a replacer channel $\mathcal{N}(\rho_{A'B'}) = \sigma_{AB}$ for all states $\rho_{A'B'}$, $f[A|B]_{\mathcal{N}} = f(A|B)_{\sigma}$.
- For $\mathcal{N}_{A'B'\to AB} = \mathcal{N}^1_{A'\to A} \otimes \mathcal{N}^2_{B'\to B}$, $f[A|B]_{\mathcal{N}} = f[A]_{\mathcal{N}^1}$.

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Quantum relative entropy: A parent information-theoretic quantity

The quantum relative entropy $D(\cdot\|\cdot)$ between

- a state ρ_A and a positive semidefinite operator σ_A is defined as

$$D(\rho_A\|\sigma_A) = -\operatorname{tr}[\rho_A \log(\rho_A - \sigma_A)] \tag{1}$$

  if $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$, else $+\infty$.

- a channel $\mathcal{N}_{A\to B}$ and a completely positive map $\mathcal{M}_{A\to B}$,

$$D[\mathcal{N}\|\mathcal{M}] := \sup_{\rho_{RA}\in \mathrm{St}(RA)} D(\mathrm{id}_R \otimes \mathcal{N}(\rho_{RA})\| \, \mathrm{id}_R \otimes \mathcal{N}(\rho_{RA})),$$

  suffices to consider optimization over pure input states.

The quantum relative entropy between

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Entropy of state

For quantum states,

- the von Neumann entropy

$$S(A)_\rho := S(\rho_A) := -D(\rho_A \| \mathbb{1}_A) = -\operatorname{tr}[\rho_A \log \rho_A]$$

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- the von Neumann conditional entropy

$$S(A|B)_\rho := -\inf_{\sigma \in \mathrm{St}(B)} D(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$$
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Entropy of channel

The replacer map $\mathcal{R}_{A'\to A}$ is a completely positive map such that $\mathcal{R}_{A'\to A}(X_{A'}) = \operatorname{tr}[X_{A'}]\mathbb{1}_A$. Its Choi operator $\Gamma^{\mathcal{R}}_{RA} := \mathrm{id}_R \otimes \mathcal{R}(\Gamma_{RA'}) = \mathbb{1}_R \otimes \mathbb{1}_A$, where $\Gamma_{RA'} = \sum_{i,j} |ii\rangle\langle jj|_{RA'}$.

The completely depolarizing channel $\widetilde{\mathcal{R}}_{A'\to A} = \frac{1}{|A|}\mathcal{R}_{A'\to A}$ always outputs maximally mixed state $\pi_A = \frac{1}{|A|}\mathbb{1}_A$ for all input states.

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For quantum channel $\mathcal{N}_{A'\to A}$, its von Neumann entropy is [GW21] (see also [SPSD24])

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Conditional entropy of a channel: Formulation

For a bipartite channel $\mathcal{N}_{A'B'\to AB}$, following definitions satisfy axioms*.

von Neumann conditional entropy

$S[A|B]_{\mathcal{N}} := -\inf_{\mathcal{M}\in\mathrm{CPTP}} D[\mathcal{N}_{A'B'\to AB}\|\mathcal{R}_{A'\to A}\otimes\mathcal{M}_{B'\to B}]$.

von Neumann NS conditional entropy

$S^{\not\to}[A|B]_{\mathcal{N}} := -D[\mathcal{N}_{A'B'\to AB}\|\mathcal{R}_{A\to A}\circ\mathcal{N}_{A'B'\to AB}]$.

*The reduced channel $\mathcal{N}_{A'B'\to A}$ of a bipartite channel $\mathcal{N}_{A'B'\to AB}$ to be defined as $\mathcal{N}_{A'B'\to A} := \mathrm{tr}_B \circ\mathcal{N}_{A'B'\to AB}$.

Then, we have, $S[A]_{\mathcal{N}} := S[\mathcal{N}_{A'B'\to A}] = -D[\mathcal{N}_{A'B'\to A}\|\mathcal{R}_{A'B'\to A}]$ and $S[A|B]_{\mathcal{N}} \leq S[A]_{\mathcal{N}}$.

Idea for channelization

The idea is to "channelize" $S(A|B)_\rho = -\inf_{\sigma\in\mathrm{St(B)}} D(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B) = -D(\rho_{AB}\|\mathbb{1}_A\otimes\rho_B)$,

$$\rho_{AB} \to \mathcal{N}_{A'B'\to AB}$$
$$\mathbb{1}_A\otimes\sigma_B \to \mathcal{R}_{A\to A}\otimes\mathcal{M}_{B'\to B}$$
$$\mathbb{1}_A\otimes\rho_B = \mathcal{R}_{A\to A}(\rho_{AB}) \to \mathcal{R}_{A\to A}\circ\mathcal{N}_{A'B'\to AB},$$

$D(\cdot\|\cdot)$ between positive operators $\to$ $D[\cdot\|\cdot]$ between completely positive maps.

Notice that $S^{\not\to}[A|B]_{\mathcal{N}} = \inf_{\psi\in\mathrm{St(RA'B')}} S(A|RB)_{\mathcal{N}(\psi)}$.

Conditional entropies & Causal structure

- For an arbitrary bipartite quantum channel $\mathcal{N}_{A'B'\to AB}$, $S[A|B]_{\mathcal{N}} \leq S^{\not\to}[A|B]_{\mathcal{N}}$.
- $S[A|B]_{\mathcal{N}} < S^{\not\to}[A|B]_{\mathcal{N}}$ iff channel $\mathcal{N}_{A'B'\to AB}$ is signaling from $A' \to B$, i.e., $\operatorname{tr}_A \circ \mathcal{N}_{A'B'\to AB} = \operatorname{tr}_{A'} \otimes \mathcal{M}_{B'\to B}$ for some $\mathcal{M} \in \mathrm{CPTP}$.
- For no-signaling $\mathcal{N}_{A'B'\to AB}$, $S[A|B]_{\mathcal{N}} = S^{\not\to}[A|B]_{\mathcal{N}}$.
- For tele-covariant channel $\mathcal{N}_{A'B'\to AB}$, $S[A|B]_{\mathcal{N}} = S(R_A A|R_B B)_{\Phi^{\mathcal{N}}} - \log|A'|$, where $\Phi^{\mathcal{N}}_{R_A A R_B B} := \mathcal{N}(\Phi_{R_A A'} \otimes \Phi_{R_B B'})$.

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- $S[A|B]_{\mathcal{N}} < S^{\not\to}[A|B]_{\mathcal{N}}$ iff channel $\mathcal{N}_{A'B'\to AB}$ is signaling from $A' \to B$, i.e., $\mathrm{tr}_A \circ \mathcal{N}_{A'B'\to AB} = \mathrm{tr}_{A'} \otimes \mathcal{M}_{B'\to B}$ for some $\mathcal{M} \in \mathrm{CPTP}$.
- For no-signaling $\mathcal{N}_{A'B'\to AB}$, $S[A|B]_{\mathcal{N}} = S^{\not\to}[A|B]_{\mathcal{N}}$.
- For tele-covariant channel $\mathcal{N}_{A'B'\to AB}$, $S[A|B]_{\mathcal{N}} = S(R_A A|R_B B)_{\Phi^{\mathcal{N}}} - \log|A'|$, where $\Phi^{\mathcal{N}}_{R_A A R_B B} := \mathcal{N}(\Phi_{R_A A'} \otimes \Phi_{R_B B'})$.

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